

# Regulator constants of integral representations of finite groups

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## Abstract

Let  $G$  be a finite group and  $p$  be a prime. We investigate isomorphism invariants of  $\mathbb{Z}_{(p)}[G]$ -lattices called regulator constants, which were originally introduced by Dokchitser–Dokchitser in the context of elliptic curves. We show that if the Sylow  $p$ -subgroups of  $G$  are cyclic, then the isomorphism class of a  $\mathbb{Z}_{(p)}[G]$ -lattice is determined by the isomorphism class of its extension of scalars to  $\mathbb{Q}$ , its regulator constants and a cohomological invariant of Yakovlev. The key step is to prove that a certain pairing, defined via regulator constants, is non-degenerate for all such groups, although it may be degenerate in general. This can be interpreted as saying that, when the Sylow  $p$ -subgroups are cyclic, regulator constants are good invariants of trivial source modules.

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# 1 Introduction

Let  $G$  be a finite group and  $p$  be any prime. We denote by  $\mathbb{Z}_{(p)}$  the localisation of  $\mathbb{Z}$  at  $p$ , and by a  $\mathbb{Z}_{(p)}[G]$ -lattice, we refer to a  $\mathbb{Z}_{(p)}[G]$ -module which is free of finite rank as a  $\mathbb{Z}_{(p)}$ -module. Consider the question “Given two  $\mathbb{Z}_{(p)}[G]$ -lattices  $M, N$ , is  $M$  isomorphic to  $N$ ?” If  $p \nmid |G|$ , it is a theorem of Maranda [CR94a, Thm. 30.16] that  $M \cong N \iff M \otimes \mathbb{Q} \cong N \otimes \mathbb{Q}$ . However, for arbitrary  $p$ , the isomorphism class of  $M \otimes \mathbb{Q}$  as a  $\mathbb{Q}[G]$ -module will, in general, no longer determine that of  $M$  as a  $\mathbb{Z}_{(p)}[G]$ -module, and we will focus on this case. We recall two additional isomorphism invariants of lattices which are useful in this context.

**Yakovlev diagrams:** Assume that  $G$  has cyclic Sylow  $p$ -subgroup  $P$ , and write  $P_i \leq P$  for the subgroup of order  $p^i$ . Yakovlev [Yak96, Thm. 2.1] showed that the diagram

$$H^1(P_0, M) \xrightarrow{\leftarrow} H^1(P_1, M) \xrightarrow{\leftarrow} \dots \xrightarrow{\leftarrow} H^1(P_r, M) \quad (\star)$$

determines the isomorphism class of a  $\mathbb{Z}_{(p)}[G]$ -lattice  $M$  up to a direct summand that has “trivial source”, i.e. a module which is a direct summand of a permutation module (cf. Section 2.2 and Theorem 5.2). Here the horizontal maps are restriction and corestriction, and each  $H^1(P_i, M)$  is considered as a module over the normaliser of  $P_i$  (cf. Theorem 5.2). We refer to  $(\star)$  as the Yakovlev diagram of  $M$ . Trivial source modules have trivial Yakovlev diagrams as, by Shapiro’s lemma, permutation modules have trivial cohomology in degree one.

**Regulator constants:** A Brauer relation in characteristic zero (resp. characteristic  $p$ ) consists of a pair of  $G$ -sets for which the associated permutation modules over  $\mathbb{Q}$  (resp.  $\mathbb{F}_p$ ) are isomorphic. Characteristic zero (resp.  $p$ ) relations form a free abelian group of finite rank, which we denote by  $BR_0(G)$  (resp.  $BR_p(G)$ ). All characteristic  $p$  relations are also characteristic zero relations so that  $BR_p(G) \subseteq BR_0(G)$  (Lemma 2.11). Each characteristic zero Brauer relation  $\theta$  has an associated isomorphism invariant of  $\mathbb{Z}_{(p)}[G]$ -lattices, called its “regulator constant”, which assigns to a  $\mathbb{Z}_{(p)}[G]$ -lattice  $M$  an element  $C_\theta(M) \in \mathbb{Q}^\times / (\mathbb{Z}_{(p)}^\times)^2$ .

In several number theoretic contexts, regulator constants have been found to both coincide with naturally occurring objects and to be computationally accessible. For example, if  $K/\mathbb{Q}$  is a Galois extension of number fields with  $G = \text{Gal}(K/\mathbb{Q})$ ,  $E/K$  is an elliptic curve and  $M = E(K)/E(K)_{\text{tors}}$ , the torsion-free quotient of the Mordell-Weil group of  $E$ , then the regulator constants of  $M \otimes \mathbb{Z}_{(p)}$  are closely related to the elliptic regulator [DD09] of  $E$ . Similarly, if  $M = \mathcal{O}_K^\times / \mu_K$  is the unit group of  $K$  modulo roots of unity, then the regulator constants of  $M \otimes \mathbb{Z}_{(p)}$  are closely related to Dirichlet’s unit group regulator [Bar12].

**Main result:** The aforementioned invariants together completely determine the isomorphism class of a  $\mathbb{Z}_{(p)}[G]$ -lattice.

**THEOREM 1.1** *Let  $G$  be a finite group and  $p$  a prime such that  $G$  has cyclic Sylow  $p$ -subgroups. Given two  $\mathbb{Z}_{(p)}[G]$ -lattices  $M, N$ , then  $M \cong N$  if and only if all the following conditions hold:*

- i)  $M \otimes \mathbb{Q} \cong N \otimes \mathbb{Q}$ ,
- ii)  $M, N$  have isomorphic Yakovlev diagrams,
- iii) for an explicit finite list of characteristic zero Brauer relations, the corresponding regulator constants of  $M, N$  are equal.

This is stated precisely as Theorem 5.4. In a way that can be made precise, i)-iii) are complementary (cf. Remark 5.5). For some groups, such as dihedral groups  $D_{2p}$  for primes  $p \leq 67$ ,

the isomorphism class of a  $\mathbb{Z}[G]$ -lattice  $M$  is determined by its localisations at the primes dividing  $|G|$ . As a result, the theorem may be applied at each prime to give data which determines the isomorphism class of  $M$  as a  $\mathbb{Z}[G]$ -lattice (cf. Remark 7.1).

**Strategy of proof:** The work of Yakovlev (cf. Theorem 5.2) reduces Theorem 1.1 to proving that, for groups with cyclic Sylow  $p$ -subgroups, regulator constants are “sufficiently good” invariants of trivial source  $\mathbb{Z}_{(p)}[G]$ -modules.

To be more specific, let  $A(\mathbb{Z}_{(p)}[G])$  denote the representation ring of  $G$  and, for any abelian group  $T$ , let  $T_{\mathbb{Q}}$  denote  $T \otimes \mathbb{Q}$ . If  $v_p(-)$  denotes the  $p$ -adic valuation, then, for any finite group  $G$ , regulator constants define a pairing

$$\begin{aligned} v_p(C_{(-)}(-)): BR_0(G)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} &\longrightarrow \mathbb{Q} \\ (\theta, M) &\longmapsto v_p(C_{\theta}(M)). \end{aligned}$$

For trivial reasons,  $BR_p(G)_{\mathbb{Q}}$  is in the left kernel of  $v_p(C_{(-)}(-))$  and modules induced from cyclic subgroups are in the right kernel (see Construction 3.1, Remark 3.2).

Let  $A(\mathbb{Z}_{(p)}[G], \text{triv})$  denote the subring of  $A(\mathbb{Z}_{(p)}[G])$  consisting of trivial source modules, and  $A(\mathbb{Z}_{(p)}[G], \text{cyc})$  be the subspace generated by permutation modules induced from cyclic subgroups. Then  $A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}/A(\mathbb{Z}_{(p)}[G], \text{cyc})_{\mathbb{Q}}$  and  $BR_0(G)_{\mathbb{Q}}/BR_p(G)_{\mathbb{Q}}$  are canonically isomorphic (Remark 3.7). The key step in the proof of Theorem 1.1 is showing that:

**THEOREM 1.2** *Given a finite group  $G$ , and prime  $p$  such that  $G$  has cyclic Sylow  $p$ -subgroup, the pairing*

$$\langle -, - \rangle_{\text{perm}}: BR_0(G)_{\mathbb{Q}}/BR_p(G)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}/A(\mathbb{Z}_{(p)}[G], \text{cyc})_{\mathbb{Q}} \rightarrow \mathbb{Q}$$

*is non-degenerate.*

This is later stated as Theorem 4.1. For arbitrary groups, the above pairing may be degenerate (for example  $C_3 \times C_3 \times S_3$  when  $p = 3$ , c.f. Section 7.2).

The proof of Theorem 1.2 relies on two observations, firstly, that we need only consider “ $p$ -hypo-elementary” groups (see Lemma 3.5, cf. Definition 2.22). Secondly, that the regulator constants of trivial source modules are controlled by those of permutation modules (cf. Theorem 2.24), which are easy to enumerate and for which we are able to give formulae for their regulator constants in terms of purely group theoretic data (Remark 3.10). Given these, Theorem 1.2 reduces to an explicit calculation.

**Further topics:** In Theorem 6.1, we prove for arbitrary non-cyclic groups that there are regulator constants which are non-trivial on specific permutation modules, so that  $v_p(C_{(-)}(-))$  is non-zero.

Finally, we briefly discuss the full pairing  $v_p(C_{(-)}(-))$  between Brauer relations and arbitrary lattices. It is interesting to ask if this pairing has zero left kernel (even when the permutation pairing need not). Whilst we may again assume that  $G$  is  $p$ -hypo-elementary, unfortunately it is unclear both how to produce  $\mathbb{Z}_{(p)}[G]$ -lattices in any generality and how to describe their regulator constants in a meaningful way.

In a future paper we intend to describe some applications of Theorem 1.1 within number theory. For example, in the case of unit groups of number fields, when  $p$  divides  $|G|$  at most once, it is possible to reinterpret all three types of invariants in terms of classical invariants of number fields.

**Outline:** In Section 2, we set out notation and recall necessary background results on Brauer relations and regulator constants. This relies on an induction theorem which should be well-known but we have been unable to find explicitly in the literature. We provide a proof of this result in

Appendix A. In Section 3, we outline precise questions on regulator constants as invariants of modules and show that they reduce to considering  $p$ -hypo-elementary groups. In Section 4, we prove Theorem 1.2, and in Section 5, we deduce from this Theorem 1.1. Separately from Sections 4 and 5, in Section 6, we prove Theorem 6.1 on regulator constants of arbitrary groups. Finally, in Section 7, we provide examples and non-examples illustrating our results.

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REMARK 1.3 The *species* of  $G$ , by which we mean ring homomorphisms  $A(\mathbb{Z}_{(p)}[G], \text{triv}) \rightarrow \mathbb{Q}$ , are well understood (cf. Theorem A.6) and give a practical method to factorise a trivial source module into indecomposables. Unfortunately, in general, species do not canonically extend to  $\mathbb{Q}$ -valued functions on the full representation ring. As such, it is not possible to talk of species of an arbitrary lattice and it is not clear that species can be combined with Yakovlev's result in any meaningful way to give a result analogous to Theorem 1.1.

In contrast, regulator constants are defined as functions on the full representation ring, but it is not obvious (or true in general) that they are good invariants of trivial source modules (cf. Theorem 4.1).

REMARK 1.4 It is essential that the coefficient ring of the lattices appearing in Theorem 1.1 is  $\mathbb{Z}_{(p)}$  as opposed to, say,  $\mathbb{Z}_p$ . This is because the lack of roots of unity restricts decomposition of permutation modules and allows us to pass from trivial source modules to permutation modules (cf. Theorem 2.24). In this way, the representation theory of  $\mathbb{Z}_{(p)}$  is closer to that of  $\mathbb{Q}$  than  $\mathbb{Z}_p$  or  $\mathbb{F}_p$ .

REMARK 1.5 It is possible to define regulator constants  $C_\theta(M)$  of a  $\mathbb{Z}[G]$ -lattice  $M$ . Then  $C_\theta(M)$  is the product of the  $p$ -part of  $C_\theta(M \otimes \mathbb{Z}_{(p)})$  for all  $p$  dividing  $|G|$  (see Remark 2.35). As a result, confining ourselves to  $\mathbb{Z}_{(p)}[G]$ -lattices over  $\mathbb{Z}[G]$ -lattices is innocuous.

## 2 Preliminaries

### 2.1 Notation

Throughout,  $G$  shall denote a finite group,  $p$  a prime and  $\mathcal{R}$  any ring, but we will be most concerned with  $\mathcal{R} = \mathbb{Z}_{(p)}$  or  $\mathbb{Q}$ .

NOTATION 2.1 We fix the following notation:

- By  $\mathbb{1}_{\mathcal{R}, G}$ , we denote the  $\mathcal{R}[G]$ -module which is free of rank 1 as an  $\mathcal{R}$ -module with trivial  $G$ -action. We frequently simply write  $\mathbb{1}_G$  or  $\mathbb{1}$  when the ring or group are clear from the context.
- Given a subgroup  $H \leq G$  and an  $\mathcal{R}[G]$ -module  $M$ , we shall denote the restriction of  $M$  to  $H$  by  $M \downarrow_H^G$  or  $M \downarrow_H$ . Similarly, given an  $\mathcal{R}[H]$ -module  $N$ , we write  $N \uparrow_H^G$  or simply  $N \uparrow^G$  for the induction of  $N$  to  $G$ .
- We say that an  $\mathcal{R}[G]$ -module is an  $\mathcal{R}[G]$ -lattice if it is free of finite rank as an  $\mathcal{R}$ -module. Let  $A(\mathcal{R}[G])$  denote the *representation ring* of  $G$ . As an abelian group,  $A(\mathcal{R}[G])$  consists of formal  $\mathbb{Z}$ -linear combinations of isomorphism classes of  $\mathcal{R}[G]$ -lattices, subject to relations of the form  $[M] + [N] = [M \oplus N]$ . Here we use  $[M]$ , or just  $M$ , to denote the element of  $A(\mathcal{R}[G])$  corresponding to an  $\mathcal{R}[G]$ -lattice  $M$ . The ring structure on  $A(\mathcal{R}[G])$  is given by

setting  $[M] \cdot [N] = [M \otimes N]$ . Induction defines a group homomorphism  $\text{ind}: A(\mathcal{R}[H]) \rightarrow A(\mathcal{R}[G])$ , whilst restriction defines a ring homomorphism  $\text{res}: A(\mathcal{R}[G]) \rightarrow A(\mathcal{R}[H])$ .

- Recall that a permutation module is a finite direct sum of modules of the form  $\mathbb{1}_H^G$  for any subgroup  $H$  of  $G$ . We denote the subgroup of  $A(\mathcal{R}[G])$  generated by such modules by  $A(\mathcal{R}[G], \text{perm})$ . The equality  $\mathbb{1}_H^G \otimes \mathbb{1}_K^G = \mathbb{1}_{H \uparrow_K^G}^G$  and Mackey's formula show that  $A(\mathcal{R}[G], \text{perm}) \subseteq A(\mathcal{R}[G])$  is a subring, which we call the *permutation ring*. Both  $\text{res}, \text{ind}$  restrict to maps of permutation rings, the former due to Mackey's formula.
- Let  $A(\mathcal{R}[G], \text{cyc})$  be the subring generated by  $\mathbb{1}_H^G$  as  $H$  runs only over cyclic subgroups.
- Given a quotient  $q: G \rightarrow G/N$  and an  $\mathcal{R}[G/N]$ -module  $M$ , we denote by  $\text{inf}_{G/N}^G(M)$  the inflation of  $M$  to  $G$ . This defines a ring homomorphism  $\text{inf}: A(\mathcal{R}[G/N]) \rightarrow A(\mathcal{R}[G])$ , which restricts to a map of permutation rings since, for  $H \leq G/N$ ,  $\text{inf}_{G/N}^G(\mathbb{1}_H^G) = \mathbb{1}_{q^{-1}(H)}^G$ .
- In the same notation, given a  $G$ -module  $M$ , we define its deflation to  $G/N$ ,  $\text{defl}_{G/N}^G M$ , to be the fixed submodule  $M^N$  with  $G/N$ -action. Restricting to permutation modules,  $(\mathbb{1}_H^G)^N \cong \mathbb{1}_{NH}^G$ , so that  $\text{defl}(\mathbb{1}_H^G) \cong \mathbb{1}_{q(H)}^{G/N}$ . The composite  $\text{defl} \circ \text{inf}$  is the identity on all of  $A(\mathcal{R}[G/N])$ .
- Given a free  $\mathbb{Z}$ -module  $B$ , we shall denote by  $B_{\mathbb{Q}}$  its basechange  $B \otimes \mathbb{Q}$ . For example,  $A(\mathcal{R}[G])_{\mathbb{Q}}$  will be used to denote  $A(\mathcal{R}[G]) \otimes_{\mathbb{Z}} \mathbb{Q}$ .
- By  $H \leq_G G$ , we denote a conjugacy class of subgroups of  $G$  with representative  $H$ . When used in indices, the symbol  $\leq_G$  denotes indexing over conjugacy classes of subgroups. Thus,  $\sum_{H \leq_G G} 1$  is equal to the number of conjugacy classes of subgroups.

**REMARK 2.2** Recall that a module is called *indecomposable* if it can not be written as a direct sum of proper submodules. If  $\mathcal{R}$  is a discrete valuation ring, then every  $\mathcal{R}[G]$ -lattice admits a unique decomposition into direct sums of indecomposables [Rei70, Cor. 5.8], so that  $A(\mathcal{R}[G])$  is free as a  $\mathbb{Z}$ -module with a basis given by isomorphism classes of indecomposable modules. If  $\mathcal{R}$  is a field of characteristic not dividing the order of  $G$ ,  $A(\mathcal{R}[G])$  is free of finite rank as an abelian group. This need not be the case in general, for example if  $G = C_2 \times C_2$  and  $\mathcal{R} = \mathbb{Z}_{(2)}$ , there are infinitely many isomorphism classes of indecomposable  $\mathbb{Z}_{(2)}[G]$ -modules, and so the representation ring has infinite rank. In general,  $A(\mathbb{Z}_{(p)}[G])$  has finite rank if and only if the Sylow  $p$ -subgroups of  $G$  are cyclic of order at most  $p^2$  [CR94a, Sec. 33].

In contrast, a generating set of  $A(\mathcal{R}[G], \text{perm})$  is given by  $\mathbb{1}_H^G$ , as  $H$  ranges over all subgroups. However, this set will rarely be linearly independent. For example,  $\mathbb{1}_H^G \cong \mathbb{1}_{H'}^G$  for conjugate  $H, H'$ . A basis of  $A(\mathbb{Z}_{(p)}[G], \text{perm})$  is exhibited by Conlon's induction theorem (see Theorem 2.24).

## 2.2 Trivial source modules

**DEFINITION 2.3** We say that a module has *trivial source*, or is a *trivial source module*, if it is a direct summand of a permutation module. We denote the subring of  $A(\mathcal{R}[G])$  generated by the trivial source modules by  $A(\mathcal{R}[G], \text{triv})$ . Since induction, restriction, inflation and deflation take permutation modules to permutation modules, there are corresponding maps for  $A(\mathcal{R}[-], \text{triv})$ .

**REMARK 2.4** When  $\mathcal{R}$  is a field of characteristic zero, all  $\mathcal{R}[G]$ -lattices are projective and thus have trivial source. The same is true if  $\mathcal{R} = \mathbb{Z}_{(p)}$  and  $p \nmid |G|$  [CR94a, Thm. 30.16]. When  $\mathcal{R}$  is a DVR, uniqueness of decomposition forces there to be finitely many indecomposable trivial source modules up to isomorphism, and  $A(\mathcal{R}[G], \text{triv})$  is free of finite rank on such modules.

## 2 Preliminaries

EXAMPLE 2.5 Let  $\mathcal{R} = \mathbb{Z}_{(p)}$  and  $G = C_p$ . Up to isomorphism, there are 3 indecomposable  $\mathbb{Z}_{(p)}[C_p]$ -lattices

$$\mathbb{1}_G, I_G, \mathbb{Z}_{(p)}[C_p],$$

the trivial module, the augmentation ideal of  $\mathbb{Z}_{(p)}[C_p]$ , and the regular representation [HR62, Thm. 2.6]. The indecomposable trivial source modules are all summands of  $\mathbb{1}_{\{1\}} \uparrow^{C_p} \cong \mathbb{Z}_{(p)}[C_p]$  or  $\mathbb{1}_{C_p} \uparrow^{C_p} = \mathbb{1}_{C_p}$ . So the trivial source indecomposables are precisely  $\mathbb{1}_G$  and  $\mathbb{Z}_{(p)}[C_p]$ , and  $A(\mathbb{Z}_{(p)}[G], \text{triv}) = A(\mathbb{Z}_{(p)}[G], \text{perm})$  is of rank 2 as a  $\mathbb{Z}$ -module.

DEFINITION 2.6 Let  $M$  be any  $\mathbb{Z}_{(p)}[G]$ -lattice. By uniqueness of decomposition, we may define  $M_{\text{triv}}$  to be the submodule generated by all indecomposable trivial source summands of  $M$ . We call  $M_{\text{triv}}$  the *trivial source part* of  $M$  and call the submodule  $M_{\text{nt}}$  generated by the non-trivial source summands the *non-trivial source part*. By definition,  $M = M_{\text{triv}} \oplus M_{\text{nt}}$ .

### 2.3 Brauer relations

DEFINITION 2.7 A  $G$ -set is a set with a left action of  $G$ . We define the *Burnside ring*  $B(G)$  of  $G$  to be the free abelian group on isomorphism classes of finite  $G$ -sets, quotiented by relations of the form  $[X \amalg Y] - [X] - [Y]$  where  $X, Y$  are any  $G$ -sets, and  $[X]$  the corresponding element of the free group. The ring structure is then given by setting  $[X] \cdot [Y] = [X \times Y]$ .

By decomposing  $G$ -sets into their orbits, we observe that  $B(G)$  is a free  $\mathbb{Z}$ -module on isomorphism classes of transitive  $G$ -sets. Every transitive  $G$ -set is of the form  $G/H$ , for some subgroup  $H \leq G$ , where  $H$  is unique up to conjugacy. We denote the element of  $B(G)$  corresponding to  $G/H$  by  $[H]$ . Thus,  $B(G)$  is free as a  $\mathbb{Z}$ -module on the set of symbols  $[H]$  for  $H \leq_G G$ .

CONSTRUCTION 2.8 For any ring  $\mathcal{R}$ , a  $G$ -set  $X$  canonically defines a permutation module and we obtain a surjective map

$$b_{\mathcal{R}}: B(G) \rightarrow A(\mathcal{R}[G], \text{perm}),$$

which sends  $[H]$  to  $\mathbb{1}_{\mathcal{R}, H} \uparrow^G$ . In the case of  $\mathcal{R} = \mathbb{Q}, \mathbb{F}_p$  we shall write  $b_0, b_p$  respectively, for  $b_{\mathcal{R}}$ . Where necessary, we record the dependence of  $b_{\mathcal{R}}$  on  $G$  by writing  $b_{\mathcal{R}, G}$  or  $b_{0, G}$ .

DEFINITION 2.9 A *Brauer relation*, of a group  $G$  over a ring  $\mathcal{R}$ , is an element of  $\ker b_{\mathcal{R}} \trianglelefteq B(G)$ . We shall refer to  $\ker b_{\mathcal{R}}$  as the space of Brauer relations over  $\mathcal{R}$  and shall denote it by  $BR_{\mathcal{R}}(G)$ .

When  $\mathcal{R} = \mathbb{Q}, \mathbb{F}_p$ , we call a Brauer relation over  $\mathcal{R}$  a relation in characteristic zero or characteristic  $p$ , respectively, and denote  $BR_{\mathcal{R}}(G)$  by  $BR_0(G), BR_p(G)$  respectively. In the literature it is common to refer to a characteristic zero relation as simply a Brauer relation, and we shall often do the same.

EXAMPLE 2.10 If  $G = S_3$ , a characteristic zero Brauer relation is given by

$$\theta: [1] + 2[G] - [C_3] - 2[C_2].$$

We shall see that in fact,  $BR_0(S_3) = \theta \cdot \mathbb{Z}$ .

All characteristic  $p$  relations are relations in characteristic zero also:

LEMMA 2.11 As subspaces of  $B(G)$ ,  $BR_p(G) = BR_{\mathbb{Z}_{(p)}}(G) \subseteq BR_0(G)$ .

*Proof.* The base change  $b_p$  factors as

$$A(\mathbb{Z}_{(p)}[G], \text{triv}) \rightarrow A(\mathbb{Z}_p[G], \text{triv}) \rightarrow A(\mathbb{F}_p[G], \text{triv}).$$

The first map is an inclusion [Rei70, Thm 5.6 iii)], whilst the second is an isomorphism [Ben98, 3.11.4 i)], so the kernels of  $b_p, b_{\mathbb{Z}_{(p)}}$  agree. Similarly, as  $b_0$  factors through  $b_{\mathbb{Z}_{(p)}}$ , there is an inclusion  $BR_{\mathbb{Z}_{(p)}}(G) \subseteq BR_0(G)$ .  $\square$

NOTATION 2.12 Let  $G$  be a finite group and  $H \leq G$  a subgroup,

- given an  $H$ -set  $X$ , we let  $X \uparrow_H^G$  denote the induced  $G$ -set  $(G \times X)/H$ ; here the  $H$ -equivalence is by acting on  $G$  on the right and  $X$  on the left, whilst  $G$  acts on the resulting set via its left action on  $G$ . For transitive  $G$ -sets  $(G/K)$  we have  $(H/K) \uparrow^G = (G/K)$  and we shall regularly abuse notation by writing  $[K] \uparrow^G$  simply as  $[K]$ , where now  $K$  is thought of as a subgroup of  $G$ ,
- if  $Y$  is a  $G$ -set, we let  $Y \downarrow_H^G$  denote its restriction to  $H$ . For a subgroup  $K$  of  $G$ , making good use of the above abuse of notation, Mackey's formula for  $G$ -sets states that

$$[K] \downarrow_H^G = \sum_{g \in K \backslash G/H} [K^g \cap H]. \quad (1)$$

If now  $N \trianglelefteq G$  with  $q: G \rightarrow G/N$  the quotient map,

- given a  $G/N$ -set  $X$ , we denote by  $\inf_{G/N}^G X$  the inflated set  $X$ , on which elements of  $G$  act via their image in the quotient. For  $H \leq G/N$ ,  $\inf_{G/N}^G([H]) = [q^{-1}(H)]$ ,
- given a  $G$ -set  $Y$ , let  $\text{defl}_{G/N}^G Y$  denote its deflation, i.e. the set  $Y^N$  with its (well defined) action of  $G/N$ . For a transitive  $G$ -set  $G/H$ , the fixed points under  $N$  is isomorphic to  $G/NH$ , which as a  $G/N$ -set is  $(G/N)/q(H)$ . In other words,  $\text{defl}_{G/N}^G([H]) = [q(H)]$ , and thus  $\text{defl} \circ \inf$  is the identity map.

All of these operations induce group homomorphisms on Burnside rings, but only restriction and inflation will in general be ring homomorphisms. Each of  $\text{ind}, \text{res}, \text{inf}, \text{defl}$  commute with  $b_{\mathcal{R}}$ . As a result, each restricts to morphisms of  $BR_{\mathcal{R}}(-)$  for any  $\mathcal{R}$ .

## 2.4 Relations in characteristic zero

Finding an explicit basis, for an arbitrary group  $G$ , of the space of Brauer relations in characteristic zero is a hard problem, which was recently completed by Bartel-Dokchitser [BD15, BD14]. On the other hand, in this section we recall that, a basis of the space  $BR_0(G)_{\mathbb{Q}}$  is provided by Artin's induction theorem.

NOTATION 2.13 Let  $\text{cyc}(G) := \{H \leq_G G \mid H\text{-cyclic}\}$  denote a set of representatives of each conjugacy class of cyclic subgroups.

THEOREM 2.14 (Artin's induction theorem [Sna94, Thm. 2.1.3]) *For any finite group  $G$  and  $\mathbb{Q}[G]$ -module  $M$ , for each cyclic  $H \leq_G G$ , there exists a unique  $\alpha_H \in \mathbb{Q}$  such that,*

$$M = \sum_{H \in \text{cyc}(G)} \alpha_H \mathbb{1}_H \uparrow^G \in A(\mathbb{Q}[G])_{\mathbb{Q}}.$$

DEFINITION 2.15 We say that an element  $\theta \in B(G)$  is supported at some set  $S$  of conjugacy classes of subgroups of  $G$  if the only  $[H]$  with non-zero coefficients lie in  $S$ .

COROLLARY 2.16 *For any finite group  $G$ ,*

- i) *the rank of  $BR_0(G)$  is equal to the number of conjugacy classes of non-cyclic subgroups of  $G$ ,*



ii) there are no non-zero characteristic zero Brauer relations supported only at cyclic subgroups.

*Proof.* Immediate from definitions.  $\square$

Note that a group  $G$  is cyclic if and only if it has no non-trivial Brauer relations.

DEFINITION 2.17 For any ring  $\mathcal{R}$  and finite group  $G$ , let  $b_{\mathcal{R},\mathbb{Q}}$  denote the base change

$$b_{\mathcal{R},\mathbb{Q}}: B(G)_{\mathbb{Q}} \rightarrow A(\mathcal{R}[G], \text{perm})_{\mathbb{Q}}.$$

We shall also call an element of the kernel of  $b_{\mathcal{R},\mathbb{Q}}$  a Brauer relation over  $\mathcal{R}$  and refer to the kernel  $BR_{\mathcal{R}}(G)_{\mathbb{Q}} = BR_{\mathcal{R}}(G) \otimes \mathbb{Q}$  the space of Brauer relations over  $\mathcal{R}$ . Where there is ambiguity, we shall refer to elements of the kernel of  $b_{\mathcal{R}}$  as *integral Brauer relations* and of  $b_{\mathcal{R},\mathbb{Q}}$  as *rational Brauer relations*.

Induction theorems of the form of Theorem 2.14 always give rise to corresponding family of (possibly rational) Brauer relations.

DEFINITION 2.18 For any group  $G$ , let

$$\mathbb{1}_G = \sum_{H \in \text{cyc}(G)} \alpha_H \mathbb{1}_H \uparrow^G,$$

where the  $\alpha_H \in \mathbb{Q}$  are given uniquely by Artin's induction theorem. Then,

$$\theta_G = [G] - \sum_{H \in \text{cyc}(G)} \alpha_H [H] \in B(G)_{\mathbb{Q}}$$

is a rational Brauer relation of  $G$ . We call  $\theta_G$  the *Artin relation* of  $G$ . Note that if  $G$  is cyclic, then  $\theta_G = 0 \in B(G)$ , otherwise  $\theta_G$  is non-zero and has  $[G]$ -coefficient 1.

For any group  $G$ , the uniqueness statement of Artin's induction theorem shows that  $\theta_G$  is the unique, up to scaling, element of  $BR_0(G)_{\mathbb{Q}}$  supported only at  $G$  and cyclic subgroups. The following example will be returned to in Section 4.2.

EXAMPLE 2.19 Let  $G$  be of the form  $C_{p^r} \rtimes C_n$ , with  $p \nmid n$ , and denote by  $S \leq C_n$  the kernel of the action  $C_n \rightarrow \text{Aut}(C_{p^r})$ . Writing  $s$  for  $|S|$ , we claim that

$$\theta_G = \frac{s}{n} \left( [S] - \frac{n}{s} \cdot [C_n] - [C_{p^r} \times S] + \frac{n}{s} [C_{p^r} \rtimes C_n] \right),$$

which can be checked by direct calculation. If the action of  $C_n$  is not faithful, then  $S$  is a non-trivial subgroup of  $G$  and quotienting by  $S$  results in a group of the same form but with faithful action. The Artin relation of  $G$  is then the inflation of the Artin relation of  $G/S$  (using that the preimage of a cyclic subgroup of  $G/S$  is a cyclic subgroup of  $G$ ).

Following Notation 2.12, when it is contextually clear we are referring to  $G$ -relations, for a subgroup  $H \leq G$ , we shall denote the  $G$  relation  $\theta_H \uparrow^G$  simply by  $\theta_H$ . Artin relations are well behaved under restriction:

LEMMA 2.20 Let  $G$  be a finite group and  $H, K$  subgroups. Then,

i) the restriction of the Artin relation of  $G$  to  $H$  is the Artin relation of  $H$ , i.e.

$$\theta_G \downarrow_H = \theta_H,$$



ii) more generally

$$\theta_H \uparrow_H^G \downarrow_K = \sum_{g \in H \backslash G / K} \theta_{H^g \cap K} \uparrow^K. \quad (2)$$

*Proof.* We prove ii). Mackey's formula (1) states that

$$[H] \downarrow_K = \sum_{g \in H \backslash G / K} [H^g \cap K].$$

Also by Mackey, for any cyclic group  $L \leq H$ ,  $[L] \uparrow_H^G \downarrow_K$  is supported at cyclic subgroups. But then  $\theta_H \uparrow_H^G \downarrow_K$  and  $\sum_{g \in H \backslash G / K} \theta_{H^g \cap K} \uparrow^K$  are two relations whose coefficients agree at all non-cyclic subgroups and since there are no relations supported at cyclic subgroups (Corollary 2.16 ii), they must therefore be equal.  $\square$

LEMMA 2.21 A basis of the space of rational Brauer relations  $BR_0(G)_{\mathbb{Q}}$  is given by the set  $\{\theta_H\}$  of Artin relations for non-cyclic  $H \leq_G G$ .

*Proof.* The  $\theta_H$  are linearly independent as each is zero on non-cyclic subgroups other than  $[H]$  and must span by Corollary 2.16 i).  $\square$

## 2.5 Relations in characteristic $p$

DEFINITION 2.22 Let  $p$  be prime. A finite group  $G$  is called  $p$ -hypo-elementary, or simply  $p$ -hypo, if  $G$  has a normal Sylow  $p$ -subgroup  $P$  and  $G/P$  is cyclic. Equivalently,  $G$  is  $p$ -hypo-elementary if it can be written in the form  $P \rtimes C_n$  for  $P$  a  $p$ -group and  $(p, n) = 1$ .

NOTATION 2.23 We denote a set  $\{H \leq_G G \mid H \text{ } p\text{-hypo}\}$  of representatives of the conjugacy classes of  $p$ -hypo-elementary subgroups by  $\text{hyp}_p(G)$ . Similarly, let  $\text{nhyp}_p(G) := \{H \leq_G G \mid H \text{ is } p\text{-hypo and non-cyclic}\}$ .

Recall that characteristic  $p$  relations coincide with  $\mathbb{Z}_{(p)}$ -relations (Lemma 2.11). The theory of rational characteristic  $p$  relations is controlled by:

THEOREM 2.24 Let  $G$  be any finite group and  $M$  a trivial source  $\mathbb{Z}_{(p)}[G]$ -module. Then, there exist unique  $\alpha_H \in \mathbb{Q}$ , as  $H$  runs over conjugacy classes of  $p$ -hypo-elementary subgroups of  $G$ , such that

$$M \cong \sum_{H \in \text{hyp}_p(G)} \alpha_H \mathbb{1}_H \uparrow^G.$$

REMARK 2.25 It is not true that  $A(\mathcal{R}[G], \text{perm})_{\mathbb{Q}} = A(\mathcal{R}[G], \text{triv})_{\mathbb{Q}}$  when  $\mathcal{R} = \mathbb{Z}_p$  or  $\mathbb{F}_p$  (take  $G = C_4$  and  $p = 5$ ). It is also not true that  $A(\mathbb{Z}_{(p)}[G], \text{triv}) = A(\mathbb{Z}_{(p)}[G], \text{perm})$  in general. For example, if  $G = C_3 \times Q_8$  and  $p \nmid 6$ , then extension of scalars induces an isomorphism  $A(\mathbb{Z}_{(p)}[G]) = A(\mathbb{Z}_{(p)}[G], \text{triv}) \cong A(\mathbb{Q}[G])$  which takes  $A(\mathbb{Z}_{(p)}[G], \text{perm})$  to  $A(\mathbb{Q}[G], \text{perm})$ . But  $A(\mathbb{Q}[G]) \neq A(\mathbb{Q}[G], \text{perm})$ .

In appendix A, we show that Theorem 2.24 follows from a standard formulation of Conlon's theorem.

COROLLARY 2.26 For any finite group  $G$ ,

- i) the rank of  $BR_p(G)$  is equal to the number of conjugacy classes of non- $p$ -hypo-elementary subgroups of  $G$ ,

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- ii) there are no non-zero characteristic  $p$  Brauer relations supported only at  $p$ -hypo-elementary subgroups.

*Proof.* Immediate.  $\square$

Note that there are no non-zero characteristic  $p$  Brauer relations for  $p$ -hypo-elementary groups, and this is only true of such groups. Analogously to before, this induction theorem gives rise to privileged relations in characteristic  $p$ .

DEFINITION 2.27 For any group  $G$  and prime  $p$ , write  $\mathbb{1}_{\mathbb{Z}_{(p)},G} = \sum_{H \in \text{hyp}_p(G)} \alpha_H \mathbb{1}_{\mathbb{Z}_{(p)},H} \uparrow_H^G$  uniquely by Theorem 2.14. Since  $BR_{\mathbb{Z}_{(p)}}(G) = BR_p(G)$  (Lemma 2.11),

$$\theta_{\text{Con},G} = [G] - \sum_{H \in \text{hyp}_p(G)} \alpha_H [H]$$

is a rational Brauer relation in characteristic  $p$  (i.e. an element of  $BR_p(G)_{\mathbb{Q}}$ ) which we refer to as the *Conlon relation* of  $G$ . Note,  $\theta_{\text{Con},G}$  is identically zero if and only if  $G$  is  $p$ -hypo-elementary. The Conlon relation is the unique  $p$ -relation supported only at  $G$  and  $p$ -hypo-elementary subgroups. However, the Conlon relation need not be unique amongst characteristic zero relations supported at these groups.

As before, when it is clear that we are referring to  $G$ -relations, for a subgroup  $H \leq G$  we denote  $\theta_{\text{Con},H} \uparrow_H^G$  simply by  $\theta_{\text{Con},H}$ . All characteristic  $p$  relations are rational linear combinations of Conlon relations:

LEMMA 2.28 Let  $G$  be a finite group and  $p$  a prime. Then,

- i) a basis of  $BR_p(G)_{\mathbb{Q}}$  is formed by the set  $\{\theta_{\text{Con},H}\}$  as  $H$  runs over conjugacy classes of non- $p$ -hypo-elementary groups,
- ii) this can be extended to a basis of  $BR_0(G)_{\mathbb{Q}}$  by adding the Artin relations  $\theta_H$  as  $H$  runs over conjugacy classes of non-cyclic  $p$ -hypo-elementary groups.

*Proof.* The proof of i) is as in Lemma 2.21. For ii), in addition use Corollaries 2.16, 2.26.  $\square$

EXAMPLE 2.29 Let  $G = D_{2p} = C_p \rtimes C_2$  be the dihedral group of order  $2p$  for  $p$  an odd prime. If  $\ell$  is any prime, then

$$\{H \leq G \mid H \text{ is } \ell\text{-hypo-elementary}\} = \begin{cases} \{1\}, C_2, C_p & \ell \neq p \\ \{1\}, C_2, C_p, D_{2p} & \ell = p \end{cases}, \quad (3)$$

and so  $\dim A(\mathbb{Z}_{(\ell)}[G], \text{perm})_{\mathbb{Q}} = A(\mathbb{Z}_{(\ell)}[G], \text{triv})_{\mathbb{Q}}$  is 3 unless  $\ell = p$  when it is 4. A basis  $S$  of  $A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}$  is formed by

$$S = \begin{cases} \mathbb{1}_{\{1\}} \uparrow^G, \mathbb{1}_{C_2} \uparrow^G, \mathbb{1}_{C_p} \uparrow^G & \ell \neq p \\ \mathbb{1}_{\{1\}} \uparrow^G, \mathbb{1}_{C_2} \uparrow^G, \mathbb{1}_{C_p} \uparrow^G, \mathbb{1}_G & \ell = p \end{cases}.$$

Since  $G$  has up to conjugacy four subgroups, of which three are cyclic,  $\text{rk } BR_0(G) = 1$ . Let  $\theta \in BR_0(G)$  be the relation

$$\theta: [1] + 2[G] - [C_p] - 2[C_2].$$

Then  $\theta = 2\theta_G$ . As  $\theta$  is indivisible as an element of  $B(G)$ , we find  $BR_0(G) = \theta \cdot \mathbb{Z}$ .

Since  $\text{rk } BR_\ell(G)$  is the number of conjugacy classes of non- $p$ -hypo-elementary subgroups, by (3),  $\text{rk } BR_\ell(G)$  is also one unless  $\ell = p$  when it is zero. Given that  $BR_\ell(G) \subseteq BR_0(G)$  (Lemma 2.11), we find

$$BR_\ell(G) = \begin{cases} \theta \cdot \mathbb{Z} & \text{if } \ell \neq p \\ 0 & \text{if } \ell = p \end{cases}.$$

The Conlon relation  $\theta_{\text{Con},G}$  is equal to the Artin relation unless  $\ell = p$  when it is zero.

## 2.6 Regulator constants

In this section we recall how to associate to a characteristic zero Brauer relation, a function on (nice)  $\mathcal{R}[G]$ -lattices called its regulator constant.

We recall the construction given in [DD09] for an arbitrary PID  $\mathcal{R}$  of characteristic zero. We are only ever concerned with  $\mathcal{R} = \mathbb{Z}, \mathbb{Z}_{(p)}$ . Let  $\mathcal{K}$  denote the field of fractions of  $\mathcal{R}$ .

**DEFINITION 2.30** An  $\mathcal{R}[G]$ -lattice  $M$  is called *rationally self-dual* if  $M \otimes \mathcal{K}$  is self-dual, i.e.  $M \otimes \mathcal{K}$  is isomorphic to its linear dual  $\text{Hom}_{\mathcal{K}}(M \otimes \mathcal{K}, \mathcal{K})$  as  $\mathcal{K}[G]$ -modules. This is equivalent to the existence of a non-degenerate  $G$ -invariant inner product on  $M \otimes \mathcal{K}$ . If an inner product on  $M \otimes \mathcal{K}$  exists, there is a restricted  $G$ -invariant inner product on  $M$ .

A rationally self-dual lattice  $M$  need not be linearly self-dual, i.e. a rationally self-dual  $M$  need not be isomorphic to  $\text{Hom}(M, \mathcal{R})$ . If  $\mathcal{R} = \mathbb{Z}, \mathbb{Z}_{(p)}$ , then, as all  $\mathbb{Q}[G]$ -modules are self-dual, all  $\mathcal{R}[G]$ -lattices are rationally self-dual.

**DEFINITION 2.31** Let  $G$  be a finite group and  $\theta = \sum_i [H_i] - \sum_j [H'_j] \in BR_0(G)$  be an integral characteristic zero Brauer relation of  $G$ . Given a rationally self-dual  $\mathcal{R}[G]$ -lattice  $M$ , fix a choice of non-degenerate  $G$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $M$ . The *regulator constant* of  $\theta$  evaluated at  $M$  is then

$$C_\theta(M) = \frac{\prod_i \det \left( \frac{1}{|H_i|} \langle \cdot, \cdot \rangle|_{M^{H_i}} \right)}{\prod_j \det \left( \frac{1}{|H'_j|} \langle \cdot, \cdot \rangle|_{M^{H'_j}} \right)} \in \mathcal{K}^\times / (\mathcal{R}^\times)^2.$$

This is independent of the choice of  $\langle \cdot, \cdot \rangle$  as an element of  $\mathcal{K}^\times / (\mathcal{R}^\times)^2$  (see [DD09, Thm. 2.17]). For  $M$  a  $\mathbb{Z}[G]$  or  $\mathbb{Z}_{(p)}[G]$ -lattice for some prime  $p$ , as we may take the pairing to be positive definite, for all characteristic zero Brauer relations  $\theta$  and modules  $M$ , we find  $C_\theta(M) > 0$ .

**REMARK 2.32** When evaluating regulator constants at the trivial module, the formula simplifies. For example, if  $\theta = \sum_i [H_i] - \sum_j [H'_j]$ , then

$$C_\theta(\mathbb{1}_G) = \frac{\prod_i \frac{1}{|H_i|}}{\prod_j \frac{1}{|H'_j|}} = \frac{\prod_j |H'_j|}{\prod_i |H_i|}. \quad (4)$$

This formula can be extended to permutation modules due to the formalism of regulator constants provided by the next lemma. That regulator constants of permutation modules can be made explicit in this way is crucial in the proof of Theorem 4.1.

### 3 Pairings from regulator constants

LEMMA 2.33 ([DD09]) *Let  $G$  be any finite group and  $H$  a subgroup and let  $\mathcal{R}, \mathcal{K}$  be as above. Assume throughout that all modules are rationally self dual. Then,*

- i) *if  $M, N$  are two  $\mathcal{R}[G]$ -lattices, then for any Brauer relation  $\theta$  of  $G$ ,  $C_\theta(M \oplus N) = C_\theta(M)C_\theta(N)$ ,*
- ii) *if  $\theta, \theta'$  are two Brauer relations for  $G$  and  $M$  any  $\mathcal{R}[G]$ -lattice, then  $C_{(\theta+\theta')}(M) = C_\theta(M)C_{\theta'}(M)$ ,*
- iii) *if  $M$  is a  $\mathcal{R}[G]$ -lattice, then  $C_{\theta \uparrow_H^G}(M) = C_\theta(M \downarrow_H^G)$ ,*
- iv) *if  $M$  is a  $\mathcal{R}[H]$ -lattice, then  $C_\theta(M \uparrow_H^G) = C_{\theta \downarrow_H^G}(M)$ ,*
- v) *if  $H$  is normal, then given a relation of  $G/H$  and a  $\mathbb{Z}_{(p)}[G]$ -lattice  $M$ ,  $C_{\inf_{G/H}^G \theta}(M) = C_\theta(\text{defl}_{G/H}^G M)$ ,*
- vi) *for any inclusion  $\mathcal{R} \hookrightarrow \mathcal{T}$ , with  $\mathcal{T}$  a PID, any relation  $\theta$ , and  $\mathcal{R}[G]$ -lattice  $M$ , we have  $C_\theta(M) = C_\theta(M \otimes \mathcal{T}) \in \mathcal{L}^\times / (\mathcal{T}^\times)^2$  where  $\mathcal{L}$  denotes the field of fractions of  $\mathcal{T}$ .*

NOTATION 2.34 By definition, regulator constants of  $\mathbb{Z}_{(p)}[G]$ -modules take values in  $\mathbb{Q}^\times / (\mathbb{Z}_{(p)}^\times)^2$ . Let  $v_p: \mathbb{Q}^\times \rightarrow \mathbb{Z}$  denote the usual  $p$ -adic valuation. This descends to a function  $\mathbb{Q}^\times / (\mathbb{Z}_{(p)}^\times)^2 \rightarrow \mathbb{Z}$  and, for any prime  $\ell \neq p$ , a function  $\mathbb{Q}^\times / (\mathbb{Z}_{(\ell)}^\times)^2 \rightarrow \mathbb{Z}/2\mathbb{Z}$ , both of which we also denote by  $v_p$ .

REMARK 2.35 Since the regulator constant of a  $\mathbb{Z}[G]$ -lattice is always a positive rational number (see Definition 2.31), Lemma 2.33 vi) shows that the regulator constant  $C_\theta(M)$  of a  $\mathbb{Z}[G]$ -lattice  $M$  is a function of the values  $v_p(C_\theta(M \otimes \mathbb{Z}_{(p)}))$  as  $p$  runs over all primes.

The following observation will be crucial:

LEMMA 2.36 ([Bar12, Lem. 3.6]) *If  $G$  is a finite group and  $\theta$  a relation in characteristic  $p$ , then for any prime  $\ell$  (possibly equal to  $p$ ) and  $M$  any  $\mathbb{Z}[G]$  or  $\mathbb{Z}_{(\ell)}[G]$ -lattice we have*

$$v_p(C_\theta(M)) = 0.$$

REMARK 2.37 If  $G$  is a finite group and  $p$  is a prime not dividing the order of  $G$ , then the  $p$ -hypo-elementary subgroups of  $G$  are the cyclic subgroups and so all characteristic zero relations are characteristic  $p$  relations (Lemma 2.28). Thus Lemma 2.36 shows that the only prime powers appearing in regulator constants divide the order of the group. If  $G$  itself is  $\ell$ -hypo-elementary, then, for  $p \neq \ell$ , all its  $p$ -hypo-elementary subgroups are cyclic and so all its characteristic 0 relations are relations in characteristic  $p$  and its regulator constants are always  $\ell^{\text{th}}$  powers.

## 3 Pairings from regulator constants

In this section, we set up the notation required to study all regulator constants at once. After making some basic observations, we are able to formulate precise questions to which our main results can be seen as partial answers. Loosely, these questions all relate to the effectiveness of regulator constants as invariants of modules.

### 3.1 The regulator constant pairing

CONSTRUCTION 3.1 Let  $G$  be any finite group and  $p$  a prime. The map

$$\begin{aligned} v_p(C_{(-)}(-)): BR_0(G) \times A(\mathbb{Z}_{(p)}[G]) &\longrightarrow \mathbb{Z} \\ (\theta, M) &\longmapsto v_p(C_\theta(M)), \end{aligned}$$

is a group homomorphism (Lemma 2.33 i), ii)). Extending  $\mathbb{Q}$ -linearly we get a pairing,

$$v_p(C_{(-)}(-)): BR_0(G)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} \longrightarrow \mathbb{Q},$$

which we also denote by  $v_p(C_{(-)}(-))$  and which we call the *regulator constant pairing*. By Lemma 2.36, this factors as

$$v_p(C_{(-)}(-)): BR_0(G)_{\mathbb{Q}}/BR_p(G)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} \longrightarrow \mathbb{Q}.$$

In Example 7.1, we calculate the full regulator constant pairing for dihedral groups  $D_{2p}$  with  $p$  odd, one of the few families of groups where a classification of all indecomposable lattices exists.

REMARK 3.2 The pairing  $v_p(C_{(-)}(-))$  is far from non-degenerate;  $A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}}$  is often infinite dimensional whilst  $BR_0(G)_{\mathbb{Q}}$  is always finite dimensional. Explicit elements of the right kernel are given by taking any lattice  $M \uparrow_H^G$  induced from a cyclic subgroup  $H$ . This pairs to zero with all relations as

$$v_p(C_{\theta}(M \uparrow_H^G)) = v_p(C_{\theta \downarrow_H^G}(M)) = 0,$$

where first equality is Lemma 2.33 iv) and the second is because cyclic groups have no non-zero Brauer relations (Corollary 2.16). The behaviour of the left kernel is less clear:

QUESTION 3.3 Are there groups for which the left kernel of  $v_p(C_{(-)}(-)): BR_0(G)_{\mathbb{Q}}/BR_p(G)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} \rightarrow \mathbb{Q}$  is non-trivial?

The following results and discussions are relevant to Question 3.3:

- If  $G$  has cyclic Sylow  $p$  subgroup, then the left kernel is trivial (see Theorem 4.1).
- For all groups  $G$ ,  $v_p(C_{(-)}(-))$  is not the zero pairing (see Theorem 6.1).
- There are several small groups for which the author has not been able to calculate the left kernel (cf. Section 7.2).

It is not obvious, even for small groups, that the regulator pairing should have trivial left kernel.

LEMMA 3.4 Let  $\theta$  be a relation of a finite group  $G$ . For any prime  $p$  the following are equivalent,

- $v_p(C_{\theta}(-)): A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} \rightarrow \mathbb{Q}$  vanishes identically,
- $v_p(C_{\theta \downarrow_H}(-)): A(\mathbb{Z}_{(p)}[H])_{\mathbb{Q}} \rightarrow \mathbb{Q}$  vanishes identically for all conjugacy classes of  $p$ -hypo-elementary subgroups  $H \leq G$ .

*Proof.* For the forward direction, use that  $v_p(C_{\theta \downarrow_H}(M)) = v_p(C_{\theta}(M \uparrow_H^G)) = 0$  (Lemma 2.33 iv)). For the reverse, write  $M = \sum \alpha_i M_i \uparrow_{H_i}^G$  with  $H_i$   $p$ -hypo-elementary, as given by Conlon's induction theorem [Ben98, Thm 5.6.8], then  $v_p(C_{\theta}(M)) = \sum \alpha_i v_p(C_{\theta}(M_i \uparrow_{H_i}^G)) = \sum \alpha_i v_p(C_{\theta \downarrow_{H_i}}(M_i)) = 0$ .  $\square$

LEMMA 3.5 For any finite group  $G$ , the following are equivalent:

- the left kernel of  $v_p(C_{(-)}(-)): BR_0(G)_{\mathbb{Q}}/BR_p(G)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} \rightarrow \mathbb{Q}$  is trivial,
- the left kernel of  $v_p(C_{(-)}(-)): BR_0(G/N)_{\mathbb{Q}}/BR_p(G/N)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[G/N])_{\mathbb{Q}} \rightarrow \mathbb{Q}$  is trivial for all normal subgroups  $N$ .

Moreover, both are implied by

- the left kernels of  $v_p(C_{(-)}(-)): BR_0(H)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[H])_{\mathbb{Q}} \rightarrow \mathbb{Q}$  are trivial for all isomorphism classes of  $p$ -hypo-elementary subgroups  $H \leq G$ .

*Proof.* To see  $i) \implies ii)$ , suppose that the left kernel of the pairing for  $G$  is trivial and let  $\theta$  be a relation for  $G/N$  which is not a relation in characteristic  $p$ . Since  $\text{defl} \circ \text{inf} = \text{id}$  and both take characteristic  $p$  relations to characteristic  $p$  relations,  $\text{inf}\theta$  must also not be a  $p$ -relation. So, by assumption, there exists an  $M$  for which  $0 \neq v_p(C_{\text{inf}\theta}(M))$ . By Lemma 2.33 v)  $v_p(C_{\text{inf}\theta}(M)) = v_p(C_\theta(\text{defl}M)) \neq 0$  and  $v_p(C_\theta(-))$  doesn't vanish identically. The reverse direction is automatic.

Now assume  $iii)$ . By Lemma 3.4,  $v_p(C_\theta(-))$  vanishes if and only if  $v_p(C_{\theta \downarrow_H}(-))$  vanishes for all  $H$ . But if  $\theta$  is not a relation in characteristic  $p$ , there exists a  $p$ -hypo-elementary subgroup  $H$  for which  $\theta \downarrow_H \neq 0$  (Lemma 2.28 i)) and so  $v_p(C_{\theta \downarrow_H}(-))$  doesn't vanish.  $\square$

## 3.2 The permutation pairing

Outside of a very few families of groups we have no good way of exhaustively describing non-trivial source modules. This severely limits our ability to make the regulator constant pairing explicit (cf. Example 7.1). On the other hand, Theorem 2.24 describes a basis of  $A(\mathbb{Z}_{(p)}[G], \text{perm})_{\mathbb{Q}}$ , and regulator constants of permutation modules are easy to calculate.

Results such as Yakovlev's Theorem 5.2 suggest we should be interested in the strength of regulator constants as invariants of trivial source modules. Since  $A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}} = A(\mathbb{Z}_{(p)}[G], \text{perm})_{\mathbb{Q}}$  (Lemma 2.24), there is no loss of information from restricting to permutation modules over trivial source modules.

**NOTATION 3.6** Let  $P(G)$  denote the free  $\mathbb{Q}$ -vector space on conjugacy classes of non-cyclic  $p$ -hypo-elementary subgroups.

**REMARK 3.7** As in Remark 3.2, if we restrict to  $A(\mathbb{Z}_{(p)}[G], \text{perm})_{\mathbb{Q}}$ , then  $v_p(C_{(-)}(-))$  factors as

$$v_p(C_{(-)}(-)): BR_0(G)_{\mathbb{Q}}/BR_p(G)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[G], \text{perm})_{\mathbb{Q}}/A(\mathbb{Z}_{(p)}[G], \text{cyc})_{\mathbb{Q}} \rightarrow \mathbb{Q}.$$

Lemma 2.28 ii) demonstrates that  $P(G)$  is canonically isomorphic to  $BR_0(G)_{\mathbb{Q}}/BR_p(G)_{\mathbb{Q}}$  (via  $H \mapsto \theta_H$ ). On the other hand, Theorem 2.14 shows that  $P(G)$  is canonically identified with  $A(\mathbb{Z}_{(p)}[G], \text{perm})_{\mathbb{Q}}/A(\mathbb{Z}_{(p)}[G], \text{cyc})_{\mathbb{Q}}$  by sending  $H$  to  $\mathbb{1}_H^G$ .

It is not true that the spaces can be identified before factoring  $v_p$ . Indeed,  $BR_0(G)$  is of dimension equal to the number of non-cyclic subgroups, whereas  $A(\mathbb{Z}_{(p)}[G], \text{perm})_{\mathbb{Q}}$  is of dimension equal to the number of non- $p$ -hypo-elementary subgroups.

**CONSTRUCTION 3.8** Via these canonical identifications, we may consider the restricted pairing of Remark 3.7 as a pairing

$$\langle , \rangle_{\text{perm}}: P(G) \times P(G) \longrightarrow \mathbb{Q},$$

sending  $(H, K)$  to  $v_p(C_{\theta_H}(\mathbb{1}_K^G))$ . We call  $\langle , \rangle_{\text{perm}}$  the *permutation pairing*.

**LEMMA 3.9** For any finite group  $G$  and prime  $p$ ,  $\langle , \rangle_{\text{perm}}: P(G) \times P(G) \longrightarrow \mathbb{Q}$  is symmetric.

*Proof.* For any two subgroups  $H$  and  $K$  of  $G$ , Lemmas 2.20, 2.33 show that

$$\begin{aligned} C_{\theta_H \uparrow^G}(\mathbb{1}_K \uparrow^G) &= C_{\theta_H \uparrow^G \downarrow_K}(\mathbb{1}_K) \\ &= \prod_{g \in H \backslash G/K} C_{\theta_{H^g \cap K}}(\mathbb{1}_{H^g \cap K}). \end{aligned}$$

Whilst,

$$\begin{aligned}
 C_{\theta_K \uparrow^G}(\mathbb{1}_H \uparrow^G) &= C_{\theta_K}(\mathbb{1}_H \uparrow^G \downarrow_K) \\
 &= C_{\theta_K}(\sum_{g \in H \backslash G/K} \mathbb{1}_{H^g \cap K} \uparrow^K) \\
 &= \prod_{g \in H \backslash G/K} (C_{\theta_{H^g \cap K}}(\mathbb{1}_{H^g \cap K})).
 \end{aligned}$$

□

REMARK 3.10 Along the same lines, Lemma 2.33 and (2) show that, for  $H, K \leq G$ ,

$$\begin{aligned}
 \langle H, K \rangle_{\text{perm}} &:= v_p(C_{\theta_H \uparrow^G}(\mathbb{1}_K \uparrow^G)) = v_p(C_{\theta_H \uparrow^G \downarrow_K}(\mathbb{1}_K)) \\
 &= \sum_{g \in H \backslash G/K} C_{\theta_{H^g \downarrow_{H^g \cap K}} \uparrow^K}(\mathbb{1}_K) \\
 &= \sum_{g \in H \backslash G/K} C_{\theta_{H^g \cap K} \uparrow^K}(\mathbb{1}_K) \\
 &= \sum_{g \in H \backslash G/K} v_p(C_{\theta_{H^g \cap K}}(\mathbb{1}_{H^g \cap K})). \tag{5}
 \end{aligned}$$

Combining this with (4) gives a formula for the permutation pairing.

Analogously to Question 3.3, it is tempting to ask if permutation pairing is non-degenerate for all groups  $G$ . This proves too naive, in Section 7.2, we exhibit a family of groups for which the permutation pairing is degenerate (e.g.  $C_3 \times C_3 \times S_3$  when  $p = 3$ ). Analogously to Lemmas 3.4, 3.5 we have:

LEMMA 3.11 Let  $\theta$  be a relation of a finite group  $G$ . For any prime  $p$  the following are equivalent,

- i)  $v_p(C_\theta(-)): A(\mathbb{Z}_{(p)}[G], \text{perm})_{\mathbb{Q}} \rightarrow \mathbb{Q}$  vanishes identically,
- ii)  $v_p(C_{\theta \downarrow_H}(-)): A(\mathbb{Z}_{(p)}[H], \text{perm})_{\mathbb{Q}} \rightarrow \mathbb{Q}$  vanishes identically for all conjugacy classes of  $p$ -hypo-elementary subgroups  $H \leq G$ .

LEMMA 3.12 For any finite group  $G$ , the following are equivalent:

- i) the permutation pairing is non-degenerate,
- ii) the permutation pairing of  $G/N$  is non-degenerate for all  $N \trianglelefteq G$ .

Moreover, both are implied by

- iii) the permutation pairing of  $H$  is non-degenerate for all  $p$ -hypo-elementary subgroups  $H$ .

The proofs are identical to 3.4, 3.5 but using Theorem 2.24. As a result we see that infinitely many groups exist where the permutation pairing is degenerate, for example, when  $p = 3$ , all groups with a  $C_3 \times C_3 \times S_3$  quotient.

The author would be very interested in an answer to the following questions:

QUESTION 3.13 Is the permutation pairing non-degenerate if and only if the left kernel of the regulator constant pairing is trivial?

Here, the forward direction is automatic.



QUESTION 3.14 Can one describe the groups for which the permutation pairing is degenerate?

Theorem 4.1 states that the permutation pairing is non-degenerate for all groups with cyclic Sylow  $p$ -subgroup. Theorem 6.1 states that for any group  $G$ , the permutation pairing is not the zero pairing.

REMARK 3.15 A related question to 3.13 is to ask about the integral structures. For example, how does the image of  $A(\mathbb{Z}_{(p)}[G], \text{triv}) \rightarrow \text{Hom}(BR_0(G)/BR_p(G), \mathbb{Q})$  compare to that of  $A(\mathbb{Z}_{(p)}[G]) \rightarrow \text{Hom}(BR_0(G)/BR_p(G), \mathbb{Q})$ ?

### 3.3 Regulator constants as invariants of trivial source modules

In this section, we observe that the isomorphism class of an arbitrary trivial source  $\mathbb{Z}_{(p)}[G]$ -modules  $M$  is determined by the isomorphism class of  $M \otimes \mathbb{Q}$  together with the values of  $v_p(C_{\theta_H}(M))$  as  $H$  runs over all non-cyclic  $p$ -hypo-elementary subgroups if and only if the permutation pairing (Construction 3.8) is non-degenerate. With this in mind, we extend the permutation pairing to take account of extension of scalars to  $\mathbb{Q}$ .

CONSTRUCTION 3.16 Let  $P''(G)$  denote the free  $\mathbb{Q}$ -vector space on conjugacy classes of cyclic subgroups. Theorem 2.14 states that there is a canonical isomorphism  $P''(G) \xrightarrow{\sim} A(\mathbb{Q}[G])_{\mathbb{Q}}$  sending  $H \rightarrow \mathbb{1}_H^G$ . In the same way,  $A(\mathbb{Z}_{(p)}[G], \text{cyc})_{\mathbb{Q}}$  is also canonically identified with  $P''(G)$ . Define a pairing

$$\begin{aligned} \langle -, - \rangle_{\text{char}} : P''(G) \times P''(G) &\longrightarrow \mathbb{Q} \\ (H, K) &\longmapsto \langle \mathbb{1}_H^G \otimes \mathbb{Q}, \mathbb{1}_K^G \otimes \mathbb{Q} \rangle_G, \end{aligned}$$

where the final inner product the usual pairing given by character theory. Then,  $\langle -, - \rangle_{\text{char}}$  is symmetric and is non-degenerate by Artin's induction theorem 2.14.

CONSTRUCTION 3.17 Let  $P'(G)$  denote the free  $\mathbb{Q}$ -vector space on conjugacy classes of  $p$ -hypo-elementary subgroups of  $G$ . We define the pairing

$$\begin{aligned} \langle -, - \rangle_* : P'(G) \times P'(G) &\longrightarrow \mathbb{Q} \\ (H, K) &\longmapsto \begin{cases} \langle \mathbb{1}_H^G \otimes \mathbb{Q}, \mathbb{1}_K^G \otimes \mathbb{Q} \rangle_{\text{char}} & \text{if } H \text{ is cyclic} \\ v_p(C_{\theta_H}(\mathbb{1}_K^G)) & \text{if } H \text{ is non-cyclic} \end{cases} \end{aligned}$$

This extends both  $\langle -, - \rangle_{\text{perm}}$  and  $\langle -, - \rangle_{\text{char}}$ .

REMARK 3.18 The pairing  $\langle -, - \rangle_*$  is chosen so that, via the identification  $P'(G) \cong A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}$  (Theorem 2.24), in the first variable, the construction extends to a pairing on the full representation ring (see Remark 5.5, cf. Remark 1.3).

LEMMA 3.19 For any finite group  $G$ , the following are equivalent,

- i) the permutation pairing of  $G$  is non-degenerate,
- ii) the pairing  $\langle -, - \rangle_*$  is non-degenerate,
- iii) the isomorphism class of an arbitrary trivial source  $\mathbb{Z}_{(p)}[G]$ -module,  $M$ , is determined by
  - a) the isomorphism class of  $M \otimes \mathbb{Q}$ , and
  - b) the valuations of the regulator constants  $v_p(C_{\theta_H}(M))$  as  $H$  runs over elements of  $\text{nchyp}_p(H)$ .

*Proof.* For equivalence of *i*) and *ii*), note that, for any cyclic subgroup  $K$ ,  $v_p(C_{\theta_H}(\mathbb{1}_K^G)) = 0$  (see Remark 3.2). Thus, if we order the canonical basis of  $P'(G)$  so that the cyclic subgroups come before the non-cyclic  $p$ -hypo-elementary subgroups, then the matrix representing  $\langle -, - \rangle_*$  is block upper triangular, whose diagonal blocks are the matrices representing  $\langle -, - \rangle_{\text{char}}$  and the permutation pairing respectively. The former is always invertible so  $\langle -, - \rangle_*$  is non-degenerate if and only if the permutation pairing is.

The equivalence of *ii*) and *iii*) follows from the canonical identifications

$$\begin{aligned} P'(G) &= A(\mathbb{Z}_{(p)}[G], \text{perm})_{\mathbb{Q}} = A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}} \\ K &\mapsto \mathbb{1}_K^G \mapsto \mathbb{1}_K^G \end{aligned}$$

given by Theorem 2.24. □

EXAMPLE 3.20 Let  $G = D_{2p}$ . Up to conjugacy, the  $p$ -hypo-elementary subgroups of  $G$  are  $S = \{\{1\}, C_2, C_p, D_{2p}\}$ . Applying (4) to  $\theta_G$  as calculated in Example 2.19, we find  $v_p(C_{\theta_G}(\mathbb{1}_G)) = -1/2$ . Thus, the matrix representing  $\langle -, - \rangle_*$  with respect to the basis of  $P'(G)$  given by  $S$  is:

$$\begin{array}{c} \{1\} \quad C_2 \quad C_p \quad D_{2p} \\ \{1\} \quad \left( \begin{array}{ccc|c} 2p & p & 2 & 1 \\ p & (p+1)/2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1/2 \end{array} \right) \\ C_2 \\ C_p \\ D_{2p} \end{array}$$

In Section 7.1, we extend this to allow arbitrary  $\mathbb{Z}_{(p)}[D_{2p}]$ -lattices.

## 4 Non-degeneracy of the permutation pairing when the $p$ -Sylow is cyclic

In this section we prove Theorem 1.2 of the introduction:

THEOREM 4.1 *Let  $G$  be a finite group and  $p$  a prime such that  $G$  has cyclic Sylow  $p$ -subgroup. Then, the permutation pairing*

$$v_p(C_{(-)}(-)): BR_0(G)_{\mathbb{Q}}/BR_p(G)_{\mathbb{Q}} \times A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}/A(\mathbb{Z}_{(p)}[G], \text{cyc})_{\mathbb{Q}} \rightarrow \mathbb{Q}$$

*is non-degenerate.*

As a result, for such groups  $G$ , the regulator constant pairing has trivial left kernel. Of particular interest is:

COROLLARY 4.2 *Let  $G$  be a finite group and  $p$  a prime for which the Sylow  $p$ -subgroup of  $G$  is cyclic. Then, the isomorphism class of a trivial source  $\mathbb{Z}_{(p)}[G]$ -module is determined by,*

- i) the isomorphism class of  $M \otimes \mathbb{Q}$ ,*
- ii) the valuations of the regulator constants  $v_p(C_{\theta_H}(M))$  as  $H$  runs over elements of  $\text{nchyp}_p(H)$ .*

*Proof.* This follows by Lemma 3.19. □

## 4.1 GCD matrices

In this section we prove a purely combinatorial statement required in the proof of Theorem 4.1. As ever, the reader is invited to skip any sections as to their tastes. Since this may be of limited independent interest this subsection is self contained.

NOTATION 4.3 For a natural number  $n$  and divisor  $s$  of  $n$ , denote by

- $D'(n)$  the set of divisors of  $n$  (ordered increasingly),
- $D(n, s) \subset D'(n)$  the set of divisors of  $n$  not dividing  $s$ ,
- $N(n)$  the symmetric matrix with rows and columns indexed by elements of  $D'(n)$  and  $(d_1, d_2)^{\text{th}}$  entry given by  $\gcd(d_1, d_2)$ ,
- $M(n, s)$  the symmetric matrix with rows and columns indexed by elements of  $D(n, s)$  and  $(d_1, d_2)^{\text{th}}$  entry given by  $(\gcd(d_1, d_2) - \gcd(d_1, d_2, s))$ .

EXAMPLE 4.4 If  $n = 12$  and  $s = 2$  then  $D(12, 2) = \{3, 4, 6, 12\}$  and

$$M(12, 2) = \begin{matrix} & \begin{matrix} 3 & 4 & 6 & 12 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 6 \\ 12 \end{matrix} & \begin{pmatrix} 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 2 & 4 & 10 \end{pmatrix} \end{matrix},$$

which has full rank.

REMARK 4.5 Matrices of the form  $N(n)$  are called GCD matrices and are always invertible (not necessarily integrally, see Lemma 4.6). Although matrices defined in a similar way to  $M(n, s)$  have been studied (see [BL89, Beg10]), we have been unable to find results in the literature that directly cover matrices of the form  $M(n, s)$ . For this reason, we have included a full calculation of their determinants and thus invertibility. First we recall the proof of the determinant formula for  $N(n)$ .

LEMMA 4.6 For any natural number  $n$ , the matrix  $N(n)$  has determinant  $\prod_{d \in D'(n)} \phi(d)$ , where  $\phi$  denotes Euler's totient function, and thus is always of full rank.

*Proof.* For  $n = p^e$  a prime power,

$$N(p^e) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & p & \dots & p \\ \vdots & \vdots & & \vdots \\ 1 & p & \dots & p^e \end{pmatrix}.$$

If  $e > 1$ , expanding the final column shows that  $\det(N(p^e)) = (p^e - p^{e-1}) \det(N(p^{e-1})) = \phi(p^e) \det(N(p^{e-1}))$ , and inductively proves the determinant formula for prime powers.

Now let  $s = rt$  with  $(r, t) = 1$ . Then, using the bijection  $D'(rt) \leftrightarrow D'(r) \times D'(t)$ , after simultaneous permutation of rows and columns (which preserves the determinant),  $N(s)$  is of the form

$$N(s) = \begin{matrix} & \begin{matrix} u_1 & u_2 & \dots & u_k \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{matrix} & \begin{array}{c|c|c|c} \gcd(u_1, u_1)N(r) & \gcd(u_1, u_2)N(r) & \dots & \gcd(u_1, u_k)N(r) \\ \gcd(u_2, u_1)N(r) & \gcd(u_2, u_2)N(r) & \dots & \gcd(u_2, u_k)N(r) \\ \vdots & \vdots & \ddots & \vdots \\ \gcd(u_k, u_1)N(r) & \gcd(u_k, u_2)N(r) & \dots & \gcd(u_k, u_k)N(r) \end{array} \end{matrix} = N(r) \otimes N(t).$$

If  $A, B$  are matrices of dimension  $m, n$  respectively, then their tensor product satisfies the familiar formula

$$\det(A \otimes B) = \det(A)^n \det(B)^m.$$

Applying this inductively, using the bijection  $D'(rt) \leftrightarrow D'(r) \times D'(t)$ ,

$$\begin{aligned} \det(N(r) \otimes N(t)) &= \left( \prod_{d \in D'(r)} \phi(d) \right)^{|D'(t)|} \cdot \left( \prod_{l \in D'(t)} \phi(l) \right)^{|D'(r)|} \\ &= \prod_{d \in D'(r)} \left( \phi(d)^{|D'(t)|} \prod_{l \in D'(t)} \phi(l) \right) \\ &= \prod_{d \in D'(r)} \prod_{l \in D'(t)} \phi(d) \phi(l) \\ &= \prod_{w \in D'(rt)} \phi(w), \end{aligned}$$

as required.  $\square$

LEMMA 4.7 *The matrix  $M(n, s)$  has full rank for all natural numbers  $n$  and divisors  $s$  of  $n$ . Moreover,  $\det(M(n, s)) = \prod_{d \in D(n, s)} \phi(d)$ , where  $\phi$  is the Euler totient function.*

*Proof.* We first prove the case when  $s = 1$ . Consider the matrix  $N(n)$  (whose determinant equals  $\prod_{d \in D'(n)} \phi(d)$  by Lemma 4.6). Within  $N(n)$ , the first row and column are constantly 1, and if we subtract the first column from all subsequent columns we get

$$\det(N(n)) = \det \left( \begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 1 & & & \\ \vdots & & & \\ 1 & & & \end{array} \begin{array}{c} \\ M(n, 1) \\ \\ \end{array} \right) = \det(M(n, 1)).$$

As  $\phi(1) = 1$ , this verifies the determinant formula in the case of  $s = 1$ .

We proceed by induction on the number of prime divisors of  $s$ . Assume that  $M(n, s)$  has determinant

$$\det(M(n, s)) = \prod_{d \in D(n, s)} \phi(d),$$

and consider  $M(p^r n, p^e s)$  with  $p \nmid n, s$ .

Let  $d$  be a divisor of  $p^r n$ , so  $d$  is of the form  $p^k d'$  with  $p \nmid d'$  and  $0 \leq k \leq r$ . Then,  $d \in D(p^r n, p^e s) \iff$  either  $k \leq e$  and  $d' \in D(n, s)$ , or  $k > e$  and  $d' \in D'(n)$ . In other words,  $D(p^r n, p^e s)$  can be partitioned as

$$D(p^r n, p^e s) = \left( \bigcup_{i=0}^e p^i D(n, s) \right) \cup \left( \bigcup_{i=p^{e+1}}^r p^i D'(n) \right).$$

Call  $D_1 = \bigcup_{i=0}^e p^i D(n, s)$  and  $D_2 = \bigcup_{i=p^{e+1}}^r p^i D'(n)$ . Simultaneously reorder the rows and columns of  $M(p^r n, p^e s)$  so that they respect this decomposition. Define  $A, B, C$  by

$$M(p^r n, p^e s) = \begin{array}{c} D_1 \\ D_2 \end{array} \left( \begin{array}{c|c} A & C \\ \hline C^T & B \end{array} \right).$$

For any two elements  $p^{l_1}d_1, p^{l_2}d_2 \in D_1$ , the corresponding entry of  $A$  is given by

$$\gcd(p^{l_1}d_1, p^{l_2}d_2) - \gcd(p^{l_1}d_1, p^{l_2}d_2, p^e s) = p^{\min\{l_1, l_2\}}(\gcd(d_1, d_2) - \gcd(d_1, d_2, s)).$$

So  $A$  is the tensor product

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & p & \dots & p \\ \vdots & \vdots & & \vdots \\ 1 & p & \dots & p^e \end{pmatrix} \otimes M(n, s) = N(p^e) \otimes M(n, s),$$

which has determinant

$$\det(A) = \det(N(p^e))^{|D(n, s)|} \cdot \det(M(n, s))^{e+1}.$$

By induction and Lemma 4.6,

$$\begin{aligned} \det(A) &= \det(N(p^e))^{|D(n, s)|} \cdot \det(M(n, s))^{e+1} \\ &= \left( \prod_{k=0}^e \phi(p^k)^{|D(n, s)|} \right) \cdot \left( \prod_{d \in D(n, s)} \phi(d)^{e+1} \right) \\ &= \prod_{k=0}^e \left( \phi(p^k)^{|D(n, s)|} \cdot \prod_{d \in D(n, s)} \phi(d) \right) \\ &= \prod_{k=0}^e \prod_{d \in D(n, s)} \phi(p^k) \phi(d) \\ &= \prod_{d \in D_1} \phi(d). \end{aligned}$$

We now row reduce to remove  $C^T$ . For  $e < k \leq r$ , let  $v_{p^k d}$  denote the row vector with entries indexed by  $D(n, s) = D_1 \cup D_2$  whose  $t^{\text{th}}$  entry is

$$\gcd(p^k d, t) - \gcd(p^k d, t, p^e s).$$

If  $p^e d$  does not divide  $s$ , then  $p^k d \in D_1$  and  $v_{p^k d}$  is the row of  $M(p^r n, p^e s)$  corresponding to  $p^k d$ . Otherwise,  $\gcd(p^k d, d_i, p^e s) = \gcd(p^e, d_i)$  and  $v_{p^k d}$  is identically zero. As a result, for  $p^k d \in D_2$ , subtracting  $v_{p^k d}$  from the  $(p^k d)^{\text{th}}$  row is an elementary row operation and preserves the rank and determinant. Call  $M'(p^r n, p^e s)$  the matrix resulting from performing this reduction for all elements of  $D_2$ . The entries of the  $p^k d^{\text{th}}$  row for  $p^k d \in D_2$  now satisfy, for  $p^{k'} d' \in D_1$ ,

$$\begin{aligned} M'(p^r n, p^e s)_{p^k d, p^{k'} d'} &= \gcd(p^k d, p^{k'} d') - \gcd(p^k d, p^{k'} d', p^e s) - \gcd(p^e d, p^{k'} d') + \gcd(p^e d, p^{k'} d', p^e s) \\ &= p^{k'} \gcd(d, d') - p^{k'} \gcd(d, d', s) - p^{k'} \gcd(d, d') + p^{k'} \gcd(d, d', s) \\ &= 0, \end{aligned}$$

and for  $p^{k'} d' \in D_2$ ,

$$\begin{aligned} M'(p^r n, p^e s)_{p^k d, p^{k'} d'} &= \gcd(p^k d, p^{k'} d') - \gcd(p^k d, p^{k'} d', p^e s) - \gcd(p^e d, p^{k'} d') + \gcd(p^e d, p^{k'} d', p^e s) \\ &= p^{\min\{k, k'\}} \gcd(d, d') - p^e \gcd(d, d', s) - p^e \gcd(d, d') + p^e \gcd(d, d', s) \\ &= (p^{\min\{k, k'\}} - p^e) \gcd(d, d'). \end{aligned}$$

Therefore, the row reduction results in a matrix of the form

$$M'(p^r n, p^e s) = \begin{matrix} & D_1 & D_2 \\ D_1 & \left( A \mid C \right) \\ D_2 & \left( 0 \mid B' \right) \end{matrix}.$$

where

$$\begin{aligned} B' &= N(n) \otimes \begin{pmatrix} p^{e+1} - p^e & p^{e+1} - p^e & \dots & p^{e+1} - p^e \\ p^{e+1} - p^e & p^{e+2} - p^e & \dots & p^{e+2} - p^e \\ \vdots & \vdots & & \vdots \\ p^{e+1} - p^e & p^{e+2} - p^e & \dots & p^r - p^e \end{pmatrix} \\ &= N(n) \otimes M(p^r, p^e) \\ &= N(n) \otimes p^e M(p^{r-e}, 1). \end{aligned}$$

Since

$$\det(M(p^r n, p^e s)) = \det(A) \cdot \det(B'),$$

to complete the proof we must show that  $\det(B') = \prod_{d \in D_2} \phi(d)$ . Indeed,

$$\begin{aligned} \prod_{d \in D_2} \phi(d) &= \prod_{k=e+1}^r \prod_{d \in D'(n)} \phi(p^k d) \\ &= \left( \prod_{k=e+1}^r \phi(p^k)^{|D'(n)|} \right) \cdot \left( \prod_{d \in D'(n)} \phi(d) \right) \\ &= \prod_{k=e+1}^r \phi(p^k)^{|D'(n)|} \det(N(n)) \\ &= \det(N(n))^{k-e} \cdot \det(M(p^k, p^e))^{|D'(n)|} \\ &= \det(B'). \end{aligned}$$

So we find

$$\det(M(p^k n, p^e s)) = \left( \prod_{d \in D_1} \phi(d) \right) \left( \prod_{d \in D_2} \phi(d) \right) = \prod_{d \in D} \phi(d).$$

This completes the proof of the determinant formula of  $M(a, b)$  by induction on the number of prime factors of  $b$ .  $\square$

## 4.2 Structure of $C_{p^k} \rtimes C_n$

In this section, we first perform an explicit calculation for  $p$ -hypo-elementary groups before deducing Theorem 4.1.

**LEMMA 4.8** *Let  $G$  be of the form  $C_{p^r} \rtimes C_n$  with  $p \nmid n$ . Further, let  $S$  denote the kernel of the action of  $C_n$  on  $C_{p^r}$ . Then, for any two subgroups  $H', K' \leq G$  of the form  $H' = C_{p^e} \rtimes H, K' = C_{p^f} \rtimes K$  with  $H, K \leq C_n \leq G$ , as elements of the Burnside ring  $B(K')$ ,*

$$\prod_{g \in H' \backslash G / K'} [H'^g \cap K'] = \frac{|C_n| |H \cap K|}{|H| |K|} [H' \cap K'] + \frac{p^{r - \max\{e, f\}} |C_n| |H \cap K \cap S|}{|H| |K|} [H' \cap K' \cap (C_{p^r} \times S)].$$

*Proof.* We first calculate the order of  $H' \backslash G / K'$ . As all  $p$ -subgroups of  $G$  are normal, there are canonical bijections

$$(C_{p^e} \rtimes H) \backslash G / (C_{p^f} \rtimes K) \leftrightarrow H \backslash G / (C_{p^{\max\{e,f\}}} \rtimes K) \leftrightarrow H \backslash ((C_{p^r} / C_{p^{\max\{e,f\}}}) \rtimes C_n) / K, \quad (6)$$

and we may assume  $e = f = 0$ . Elements of  $H \backslash G / K$  are in bijection with  $H$ -orbits of cosets  $gK$ . For such  $G$ , a set of coset representatives of  $G/K$  is given by elements  $\sigma\tau_i$ , where  $\sigma \in C_{p^r}$  and  $\{\tau_i\}$  are a set of coset representatives of  $C_n/K$ . The stabilizer of a right coset  $gK$  under the action of  $H$  is given by

$$\text{Stab}_H(gK) = H \cap gKg^{-1}.$$

Using that  $C_n$  is abelian, for  $k \in K$ ,

$$\begin{aligned} (\sigma\tau_i)k(\sigma\tau_i)^{-1} &= \sigma\tau_i k \tau_i^{-1} \sigma^{-1} = \sigma k \sigma^{-1} \\ &= \sigma k \sigma^{-1} k^{-1} k = \sigma \varphi(k) (\sigma^{-1}) k, \end{aligned}$$

where  $\varphi: C_n \rightarrow \text{Aut}(C_{p^r})$  denotes the action of conjugation. Since the prime to  $p$ -part of  $\text{Aut}(C_{p^r})$  equals that of  $\text{Aut}(C_{p^e})$  for any non-trivial  $C_{p^e} \leq C_{p^r}$ ,  $k$  acts trivially on  $\sigma \neq e$  if and only if  $k \in S$ . Thus,

$$\sigma \varphi(k) (\sigma^{-1}) k \in H \iff k \in H \text{ and } k \in \begin{cases} K & \text{if } \sigma = e \\ K \cap S & \text{if } \sigma \neq e \end{cases}.$$

In particular,

$$\text{Stab}_H(gK) = \begin{cases} H \cap K & \text{if } g \in C_n \\ H \cap K \cap S & \text{if } g \notin C_n \end{cases}.$$

By orbit-stabiliser theorem, there are  $\frac{|C_n||H \cap K|}{|H||K|}$  double cosets  $HgK$  of length  $\frac{|H||K|}{|H \cap K|}$  and  $(p^r - 1) \frac{|C_n||H \cap K \cap S|}{|H||K|}$  double cosets of length  $\frac{|H||K|}{|H \cap K \cap S|}$ . Furthermore, as  $H$  has a unique subgroup of each order

$$H^g \cap K = \begin{cases} H \cap K & \text{if } g \in C_n \\ H \cap K \cap S & \text{else} \end{cases},$$

and applying Mackey's formula (1) p7 gives the desired formula in this case.

If now  $e, f \geq 0$ , taking preimages under the canonical bijections (6), we find there are  $\frac{|C_n||H \cap K|}{|H||K|}$  double cosets of length  $\frac{|H||K|}{|H \cap K|} p^{\max\{e,f\}}$  and  $(p^{\max\{e,f\}} - 1) \frac{|C_n||H \cap K \cap S|}{|H||K|}$  double cosets of length  $\frac{|H||K|}{|H \cap K \cap S|} p^{\max\{e,f\}}$ . Taking preimages,

$$H'^g \cap K' = \begin{cases} H' \cap K' & \text{if } g \in C_n \\ H' \cap K' \cap (C_{p^r} \times S) & \text{else} \end{cases}.$$

Therefore, indeed

$$\coprod_{g \in H' \backslash G / K'} [H'^g \cap K'] = \frac{|C_n||H \cap K|}{|H||K|} [H' \cap K'] + \frac{p^{r-\max\{e,f\}} |C_n||H \cap K \cap S|}{|H||K|} [H' \cap K' \cap (C_{p^r} \times S)].$$

□



*Proof of Theorem 4.1.  $G$  is  $p$ -hypo-elementary:* Assume that  $G$  is  $p$ -hypo-elementary, i.e.  $G \cong C_{p^r} \rtimes C_n$  with  $p \nmid n$ . For notational convenience, we make a fixed choice of subgroup of  $G$  isomorphic to  $C_n$ , which we also denote by  $C_n$ . Let  $S$  denote the kernel of the map  $C_n \rightarrow \text{Aut}(C_{p^r})$  defining the semi-direct product. Note that  $S$  is also the kernel of the map  $C_n \rightarrow \text{Aut}(C_{p^k})$  for all  $1 \leq k \leq r$ . Up to conjugacy, any subgroup of  $G$  is of the form  $C_{p^k} \rtimes L$ , with  $L$  contained in the fixed choice of  $C_n$ . Moreover, such a subgroup is cyclic and normal in  $G$  if and only if  $L \leq S$ .

Let  $H', K'$  be non-cyclic subgroups of  $G$ . We may assume, by replacing  $H', K'$  with conjugate subgroups if necessary, that  $H' = C_{p^e} \rtimes H, K' = C_{p^f} \rtimes K$  with  $H, K \leq C_n$ . We first calculate  $\langle H', K' \rangle_{\text{perm}} = v_p(C_{\theta_{H'}}(\mathbb{1}_{K'} \uparrow^G)) = v_p(C_{\theta_{H'} \downarrow_{K'}}(\mathbb{1}_{K'}))$ . The decomposition of  $\theta_{H' \downarrow_{K'}}$  matches that of its leading term (Lemma (2.20)), so applying Lemma 4.8 we find

$$\theta_{H' \uparrow^G \downarrow_{K'}} = \left( \frac{|C_n||H \cap K|}{|H||K|} \right) \cdot \theta_{H' \cap K' \uparrow^{K'}} + \left( \frac{p^{r-\max\{e,f\}}|C_n||H \cap K \cap S|}{|H||K|} \right) \cdot \theta_{H' \cap K' \cap (C_{p^r} \times S) \uparrow^{K'}}.$$

But  $H' \cap K' \cap (C_{p^r} \times S)$  is cyclic (so that  $\theta_{H' \cap K' \cap (C_{p^r} \times S)} = 0$ ) and we find

$$v_p(C_{\theta_{H'}}(\mathbb{1}_{K'} \uparrow^G)) = \frac{|C_n||H \cap K|}{|H||K|} v_p(C_{\theta_{H' \cap K'}}(\mathbb{1}_{H' \cap K'})).$$

Let  $L'$  be an arbitrary non-cyclic subgroup of the form  $C_{p^\ell} \rtimes L$  with  $L \leq C_n$ . Directly applying (4) to the formula of Example 2.19, or by looking ahead to Example 6.18, we find that

$$\begin{aligned} v_p(C_{\theta_{L'}}(\mathbb{1}_{L'})) &= -\ell(1 - \frac{|Z_{L'}(C_{p^\ell})|}{|N_{L'}(C_{p^\ell})|}) \\ &= -\ell(1 - \frac{|L \cap S|}{|L|}). \end{aligned}$$

Concluding our calculation of  $\langle H', K' \rangle_{\text{perm}}$ , we find

$$\begin{aligned} \langle H', K' \rangle_{\text{perm}} &= \frac{|C_n||H \cap K|}{|H||K|} v_p(C_{\theta_{H' \cap K'}}(\mathbb{1}_{H' \cap K'})) \\ &= \frac{|C_n||H \cap K|}{|H||K|} \min\{e, f\} \left( \frac{|H \cap K \cap S|}{|H \cap K|} - 1 \right) \\ &= \frac{|C_n| \min\{e, f\}}{|H||K|} (|H \cap K \cap S| - |H \cap K|). \end{aligned} \tag{7}$$

Let  $T$  be the matrix representing the pairing  $\langle -, - \rangle_{\text{perm}}$  with respect to the basis of  $P(G)$  given by non-cyclic  $p$ -hypo-elementary subgroups ordered (totally in our case) by size. After a non-zero scaling of the rows and columns of  $T$ , we obtain a matrix  $T'$  with  $(H', K')^{\text{th}}$  entry

$$T'_{H', K'} = \min\{e, f\} (|H \cap K \cap S| - |H \cap K|).$$

Note  $T'$  remains symmetric and has the same rank as  $T$ . Since  $C_n$  is cyclic,  $|H \cap K \cap S| = \gcd(|H|, |K|, |S|)$  and  $|H \cap K| = \gcd(|H|, |K|)$ . Thus,  $T'$  is the matrix with entries

$$T'_{H', K'} = \min\{e, f\} (\gcd(|H|, |K|, |S|) - \gcd(|H|, |K|)). \tag{8}$$

Let  $M(m, l)$  be as in Notation 4.3. If  $Q(d)$  denotes the  $d \times d$  matrix with  $Q_{i,j} = \min\{i, j\}$ , then, by (8), we may simultaneously permute the rows and columns of  $T'$  to get

$$T' \sim -Q(r) \otimes M(n, s),$$

where  $|S| = s$ . As  $Q(r)$  is manifestly of full rank and Lemma 4.7 states that  $M(n, s)$  is also, so must  $T$  be.

**Arbitrary  $G$ :** Lemma 3.12 states that the permutation pairing for  $G$  is non-degenerate if the pairing is non-degenerate for all  $p$ -hypo-elementary subgroups. So we are reduced to the above calculation.  $\square$

EXAMPLE 4.9 Let  $G = C_7 \rtimes C_{12}$ . A set of representatives of the non-cyclic conjugacy classes of  $G$  is given by

$$S := \{C_7 \rtimes C_3, C_7 \rtimes C_4, C_7 \rtimes C_6, C_7 \rtimes C_{12}\}.$$

Applying (7), the matrix  $T$  representing the permutation pairing with respect to the basis given by  $S$  is given by

$$\begin{pmatrix} -8/3 & 0 & -4/3 & -2/3 \\ 0 & -3/2 & 0 & -1/2 \\ -4/3 & 0 & -4/3 & -2/3 \\ -2/3 & -1/2 & -2/3 & -5/6 \end{pmatrix}.$$

In the notation of the proof of Theorem 4.1,  $n = 12$  and  $s = 2$ . After rescaling the rows and columns of  $T$  as in the proof, we obtain the matrix  $M(12, 2)$  of Example 4.4.

## 5 Invariants of lattices for groups with cyclic Sylow subgroup

Throughout, let  $G$  be a finite group and  $p$  a prime such that  $G$  has cyclic Sylow  $p$ -subgroup. For such groups we showed in Corollary 4.2 that, the isomorphism class of a trivial source  $\mathbb{Z}_{(p)}[G]$ -module is determined by its extension of scalars to  $\mathbb{Q}$  and the value of its regulator constants. In this section we show how to combine this with existing results to provide a list of invariants which determine the isomorphism class of an arbitrary  $\mathbb{Z}_{(p)}[G]$ -lattice.

NOTATION 5.1 Let  $P$  be a choice of Sylow  $p$ -subgroup of  $G$ . Let  $r$  be such that  $P \cong C_{p^r}$ , and for  $0 \leq i \leq r$  let  $P_i \leq P$  denote the subgroup of order  $p^i$ .

Note that for a  $\mathbb{Z}_{(p)}[G]$ -lattice  $M$ ,  $H^1(P_i, M)$  is a  $N_G(P_i)$ -module. Recall that given a  $\mathbb{Z}_{(p)}[G]$ -lattice  $M$ , we define  $M_{\text{triv}}, M_{\text{nt}}$  to be the trivial source and non-trivial source parts of  $M$  respectively (see Definition 2.6)

THEOREM 5.2 (Yakovlev [Yak96, Thm. 2.1]) *Let  $G$  be a finite group and  $p$  a prime such that  $G$  has cyclic Sylow  $p$ -subgroup. If  $M$  is a  $\mathbb{Z}_p[G]$ -lattice, then the isomorphism class of  $M_{\text{nt}}$  is determined by the following diagram,*

$$H^1(P_r, M) \begin{array}{c} \xrightarrow{\text{res}^*} \\ \xleftarrow{\text{cores}_*} \end{array} H^1(P_{r-1}, M) \begin{array}{c} \xrightarrow{\text{res}^*} \\ \xleftarrow{\text{cores}_*} \end{array} \dots \begin{array}{c} \xrightarrow{\text{res}^*} \\ \xleftarrow{\text{cores}_*} \end{array} H^1(P_0, M).$$

Figure 1: Yakovlev diagram

To be precise, when we say determined by we mean that, if  $M'$  is another  $\mathbb{Z}_{(p)}[G]$ -lattice for which there are  $\mathbb{Z}_p[N_G(P_i)]$ -module isomorphisms  $\kappa_i : H^1(P_i, M) \rightarrow H^1(P_i, M')$ ,  $0 \leq i \leq n$  which commute with restriction and corestriction in the above diagram, then  $M_{\text{nt}} \cong M'_{\text{nt}}$ .

Note that Yakovlev's theorem can be seen as a partial converse to the observation that trivial source modules have trivial cohomology in degree one.

CONSTRUCTION 5.3 Call any diagram of the form

$$\bullet \xrightleftharpoons[b_r]{a_r} \bullet \xrightleftharpoons[b_{r-1}]{a_{r-1}} \dots \xrightleftharpoons[b_1]{a_1} \bullet,$$

with the  $i^{\text{th}}$  term a finite  $N_G(P_{r-i+1})$ -module and  $a_i, b_i$  homomorphisms of abelian groups, a *Yakovlev diagram*. For any  $M$ , Figure 1 is of this form and we refer to it as the *Yakovlev diagram of  $M$* . Not all such diagrams need arise as Yakovlev diagrams for some  $M$ .

There is an obvious notion of direct sum of such diagrams. Let  $\mathcal{C}$  denote the free  $\mathbb{Q}$ -vector space on isomorphism classes of such diagrams subject to identifying addition of diagrams with addition of elements of  $\mathcal{C}$ . Taking Yakovlev diagrams defines a canonical map

$$\text{Yak}: A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} \rightarrow \mathcal{C}.$$

Yakovlev's theorem is now the assertion that  $\ker(\text{Yak}) = A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}$ .

We are now able to correctly formulate the theorem outlined in the introduction:

**THEOREM 5.4** *Let  $G$  be any finite group and  $p$  a prime such that  $G$  has cyclic Sylow  $p$ -subgroup. Then, the isomorphism class of any  $\mathbb{Z}_{(p)}[G]$ -lattice  $M$  is determined by*

- i) *the isomorphisms class of  $M \otimes \mathbb{Q}$  as a  $\mathbb{Q}[G]$ -module,*
- ii) *the valuations  $v_p(C_{\theta_H}(M))$  of regulator constants of Artin relations for  $H \in \text{nychp}_p(G)$ ,*
- iii) *the Yakovlev diagram*

$$H^1(P_r, M) \xrightleftharpoons[\text{cores}_*]{\text{res}^*} H^1(P_{r-1}, M) \xrightleftharpoons[\text{cores}_*]{\text{res}^*} \dots \xrightleftharpoons[\text{cores}_*]{\text{res}^*} H^1(P_0, M)$$

as an element of  $\mathcal{C}$ .

*Proof.* Suppose  $M$  lies in the kernel of  $\text{Yak}_G$ , i.e.  $M$  is trivial source. Then, the data of i), ii) by Corollary 4.2 determine  $M$ . Thus together i), ii), iii) determine the isomorphism class of any  $\mathbb{Z}_{(p)}[G]$ -module.  $\square$

**REMARK 5.5** Recall that  $P(G), P'(G), P''(G)$  are defined to be the free  $\mathbb{Q}$ -vector spaces on conjugacy classes of non-cyclic  $p$ -hypo-elementary subgroups of  $G$ ,  $p$ -hypo-elementary subgroups and cyclic subgroups respectively. In the proof of Lemma 3.19, we showed that if we order the canonical basis of  $P'(G)$  so that the cyclic subgroups come first, then the pairing  $\langle -, - \rangle_*$  is represented by a matrix  $E$  which is block upper triangular.

We can extend this observation in light of Yakovlev's theorem. Recall that  $P'(G)$  canonically identifies with  $A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}$  via  $K \mapsto \mathbb{1}_K^G$ . As a result,  $\langle -, - \rangle_*$  extends to a pairing  $P'(G) \times A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} \rightarrow \mathbb{Q}$  by setting

$$(H, M) \mapsto \begin{cases} \langle \mathbb{1}_H^G \otimes \mathbb{Q}, M \rangle_{\text{char}} & \text{if } H \text{ is cyclic} \\ v_p(C_{\theta_H}(M)) & \text{if } H \text{ is non-cyclic} \end{cases}$$

and there is a corresponding map  $\kappa: A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} \rightarrow P'(G)^{\vee}$ .

Consider the map  $\kappa \oplus \text{Yak}: A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}} \rightarrow P'(G)^{\vee} \oplus \mathcal{C}$ . Give  $P'(G)^{\vee}$  its canonical dual basis, ordered so that cyclic subgroups come first. We saw that a basis of  $A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}$  is

given by  $\{\mathbb{1}_H^G\}_{H \in \text{hyp}_p(G)}$  (Theorem 2.14). If we give  $A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}}$  any basis extending that of  $A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}$ , then  $\kappa \oplus \text{Yak}$  is represented by a matrix of the form

$$\begin{array}{c} P''(G) \\ P(G) \\ \mathcal{C} \end{array} \left( \begin{array}{c|c|c} A & * & * \\ \hline 0 & T & * \\ \hline 0 & 0 & F \end{array} \right).$$

The vertical lines separate  $\kappa \oplus \text{Yak}$  on  $A(\mathbb{Z}_{(p)}[G], \text{cyc})_{\mathbb{Q}}$ , its extension to  $A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}$  and its extension to all of  $A(\mathbb{Z}_{(p)}[G])_{\mathbb{Q}}$ . Here  $A$  is symmetric and represents  $\langle -, - \rangle_{\text{char}}$ ,  $T$  is symmetric and represents  $\langle -, - \rangle_{\text{perm}}$ . The top left four blocks represent  $\langle -, - \rangle_*$ . These are of finite size but the remaining blocks will often be infinite (cf. Remark 2.2). The block  $F$  need not be square but Yakovlev's Theorem states that  $F$  has full column rank. The upper zero is due to Lemma 2.36 and the lower zeroes are the statement that trivial source modules have trivial cohomology in degree one.

## 6 Non-vanishing of the Artin regulator constant

In this section we prove:

**THEOREM 6.1** *For any finite group  $G$  and prime  $p$ ,  $v_p(C_{\theta_G}(\mathbb{1}_G)) \neq 0$  if and only if  $G$  contains a non-cyclic  $p$ -hypo-elementary subgroup. If  $G$  does contain a non-cyclic  $p$ -hypo-elementary subgroup then  $v_p(C_{\theta_G}(\mathbb{1}_G)) \leq -p/|G|$ . Here,  $\mathbb{1}_G$  denotes the trivial  $\mathbb{Z}_{(p)}[G]$ -module.*

The method of proof of Theorem 6.1 is disjoint to Sections 4 and 5, and is of explicit group theoretic nature.

**REMARK 6.2** The forward direction of 6.1 is formal: If  $G$  contains no non-cyclic  $p$ -hypo-elementary groups then all characteristic zero relations are relations in characteristic  $p$  (cf. Lemma A.8). But the regulator constant of a characteristic  $p$  relation has trivial valuation at  $p$  when evaluated at any lattice (Lemma 2.36).

**REMARK 6.3** Let  $G$  be a  $p$ -hypo-elementary group. Then, in terms of the permutation pairing of Construction 3.8, the theorem asserts that every entry in the row and column corresponding to  $G$  is strictly negative. By Lemma 3.5, the regulator constant pairing is non-degenerate whenever each  $p$ -hypo-elementary subgroup of  $G$  contains only cyclic proper subgroups, e.g.  $G = S_4$ . Under the same hypothesis, trivial source  $\mathbb{Z}_{(p)}[G]$ -modules are determined by extension of scalars to  $\mathbb{Q}$  and regulator constants (Lemma 3.19).

**COROLLARY 6.4** *For any finite group  $G$ , as a function on  $\mathbb{Z}[G]$ -modules, the regulator constant associated to the Artin relation  $\theta_H$  vanishes identically if and only if  $H$  is cyclic.*

*Proof.* The reverse direction is clear, as for cyclic groups  $\theta_H = 0$ . Now suppose  $H$  is non-cyclic.

**CLAIM:** A finite group is cyclic if and only if all its  $\ell$ -hypo-elementary subgroups are cyclic for every  $\ell$ .

Assuming this, the result follows from Lemmas 2.20 i), 2.33 iv) and the observation that for a  $\mathbb{Z}[G]$ -module  $M$   $v_p(C_{\theta}(M)) = v_p(C_{\theta}(M \otimes \mathbb{Z}_{(p)}))$  (see Lemma 2.33 vi)).

For the claim, the forwards direction is trivial so assume  $T$  is a group for which all  $\ell$ -hypo-elementary subgroups are cyclic for all  $\ell$ . Then, every Sylow subgroup's normaliser is equal to

its centraliser. Burnside's normal  $p$ -complement theorem then forces each Sylow  $p$ -subgroup to normalise every Sylow  $\ell$ -subgroup for  $\ell \neq p$ . As a result,  $T$  is direct product of its cyclic Sylow subgroups and is thus cyclic.  $\square$

REMARK 6.5 By symmetry (Lemma 3.9), we find that a permutation module  $\mathbb{1} \uparrow_H^G$  is trivial under all regulator constants if and only if it is cyclic.

## 6.1 Explicit Artin induction

The proof of Theorem 6.1 is made possible by Brauer's formula for explicit Artin induction.

NOTATION 6.6 Let  $\mu(n)$  denote the Möbius function of a natural number  $n$ ,

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is squarefree and has } r \text{ distinct prime factors} \\ 0 & \text{if } n \text{ is not squarefree} \end{cases}.$$

Note that  $\mu(1) = 1$ .

LEMMA 6.7 (Brauer, [Sna94, Thm. 2.1.3]) *If  $G$  is any finite group and  $\theta_G: [G] - \sum_{\text{cyc}(G)} \alpha_H[H]$  is its Artin relation, then*

$$\alpha_H = \frac{1}{|N_G(H) : H|} \sum_{C \geq H} \mu(|C : H|).$$

Here the sum runs over all cyclic overgroups of  $H$  (not just up to conjugacy).

LEMMA 6.8 *Let  $G$  be a  $p$ -hypo-elementary group and  $\theta_G = [G] - \sum_{\substack{H \leq G \\ H \text{ cyclic}}} \alpha_H[H]$ . Then  $\alpha_H \in \frac{p}{|G|} \cdot \mathbb{Z}$ .*

*Proof.* Let  $G = P \rtimes C$  be non-cyclic and  $H \leq S$ . By explicit Artin induction,  $\alpha_H \in \frac{1}{|N_G(H) : H|} \cdot \mathbb{Z}$ , so there is only anything to prove when  $H$  is of order coprime to  $p$  and  $H$  is normalised by  $P$  (and so by  $G$ ). Such an  $H$  must therefore lie in the kernel  $S$  of the action of  $C$  on  $P$ .

Let  $q$  be the quotient map  $q: G \rightarrow G/H$ . Then, a subgroup  $K \leq G$  is cyclic if and only if  $q(K)$  is. So  $q$  defines an index preserving bijection between cyclic subgroups of  $G$  containing  $H$  and cyclic subgroups of  $G/H$ . As such we, without loss of generality, assume that  $H = \{1\}$ .

We shall show that the contributions to  $\sum_{K \text{ cyclic}} \mu(|K|)$  from cyclic subgroups of order coprime to  $p$ , and of order divisible by  $p$  exactly once, cancel. Let  $K$  be a cyclic subgroup of  $G$  of order coprime to  $p$ . We split into two cases: First assume  $K$  is normal. Any cyclic group containing  $K$  with index  $p$  is of the form  $C_p \times K$  for some  $C_p \leq P$ . By (the general form of) Sylow's theorems there are  $1 \pmod{p}$  such choices. Since  $\mu(|C_p \times K|) = -\mu(|K|)$  the contributions of  $K$  and its overgroups cancel modulo  $p$ .

Now assume that  $K$  is not normal. In particular,  $K$  is not normalised by  $P$  and there are no cyclic subgroups isomorphic to  $C_p \times K$ . As  $P$  acts transitively on the non-singleton set of conjugates of  $K$ , orbit-stabiliser shows that the number of subgroups of  $G$  isomorphic to  $K$  is  $0 \pmod{p}$ . We have exhausted all cyclic subgroups and thus  $p$  divides  $\sum_{\substack{C \leq G \\ C \text{ cyclic}}} \mu(|C|)$  and  $\alpha_H \in \frac{p}{|G|} \cdot \mathbb{Z}$ .  $\square$

COROLLARY 6.9 *For any non-cyclic  $p$ -hypo-elementary group  $G$  and module  $M$ ,  $v_p(C_{\theta_G}(M)) \in \frac{p}{|G|} \cdot \mathbb{Z}$ . More generally, for any finite group  $G$ , given subgroups  $H, K$  and a  $K$  module  $M$ ,  $v_p(C_{\theta_H}(M \uparrow_K^G)) \in \frac{p}{\gcd\{|H|, |K|\}} \cdot \mathbb{Z}$ .*

*Proof.* By definition the valuations of regulator constants of integral Brauer relations lie in  $\mathbb{Z}$ , so the first statement follows from the lemma and 2.33 iii). For the second, the formalism of Lemma 2.33 and Mackey's formula gives

$$\begin{aligned} v_p(C_{\theta_H \uparrow_H^G}(M \uparrow_K^G)) &= v_p(C_{\theta_H}(M \uparrow_K^G \downarrow_H)) \\ &= \sum_{g \in H \backslash G/K} v_p(C_{\theta_H}(M^g \downarrow_{K^g \cap H} \uparrow^H)) \\ &= \sum_{g \in H \backslash G/K} v_p(C_{\theta_H \downarrow_{K^g \cap H}}(M^g \downarrow_{K^g \cap H})). \end{aligned}$$

But applying the first statement, each term of the sum lies in  $\frac{p}{\gcd\{|H|, |K|\}} \cdot \mathbb{Z}$ . □

We now look to use explicit Artin induction to provide a formula for  $v_p(C_{\theta_G}(\mathbb{1}_G))$ .

NOTATION 6.10 Recall that if two subgroups  $H_1, H_2$  of  $G$  are conjugate, then  $[H_1]$  and  $[H_2]$  are isomorphic as  $G$ -sets. To make the Artin relation slightly more canonical, instead of writing

$$\theta_G = [G] - \sum_{H \leq G} \alpha_H [H],$$

we can choose to write  $\theta_G$  uniquely as

$$\theta_G = [G] - \sum_{H \leq G} \alpha'_H [H],$$

subject to the stipulation that  $\alpha'_{H_1} = \alpha'_{H_2}$  for conjugate  $H_1, H_2$ . Then  $\alpha'_H = \frac{1}{|G:N_G(H)|} \cdot \alpha_H$ , the  $\alpha'_H$  are unique and the two notational choices denote identical elements of  $B(G)$ .

NOTATION 6.11 Let  $\mathbb{1}_p(-)$  denote characteristic function:

$$\mathbb{1}_p(-): G \rightarrow \{0, 1\}, \quad g \mapsto \begin{cases} 1 & \text{if } |g| \text{ is divisible by } p \\ 0 & \text{else} \end{cases}.$$

LEMMA 6.12 For any group  $G$  and prime  $p$ , if  $\theta_G$  denotes the Artin relation, then

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -v_p(|G|) + \frac{1}{|G|} \sum_{g \in G} v_p(|g|) + \frac{1}{|G|} \sum_{g \in G} \frac{\mathbb{1}_p(g)}{p-1}. \quad (9)$$

*Proof.* Running over all cyclic subgroups rather than their conjugacy classes, explicit Artin induction gives that

$$\theta_G = [G] - \sum_{\substack{H \leq G \\ H \text{-cyclic}}} [H] \cdot \frac{1}{|G:H|} \sum_{\substack{C \geq H \\ C \text{-cyclic}}} \mu(|C:H|).$$

Applying the formula (4) for regulator constants at the trivial module we find that

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -v_p(|G|) + \sum_{H \leq G} \frac{v_p(|H|)}{|G:H|} \sum_{C \geq H} \mu(|C:H|),$$

where from now on it is assumed that sums run only over all cyclic subgroups or overgroups. Changing the order of summation,

$$\begin{aligned} v_p(C_{\theta_G}(\mathbb{1}_G)) &= -v_p(|G|) + \sum_{C \leq G} \sum_{H \leq C} \frac{v_p(|H|)}{|G:H|} \mu(|C:H|) \\ &= -v_p(|G|) + \sum_{\substack{C \leq G \\ p \nmid |C|}} \sum_{H \leq C} \frac{v_p(|H|)}{|G:H|} \mu(|C:H|) \end{aligned}$$

as only subgroups  $C$  for which  $p$  divides  $|C|$  make any contribution. Within the second sum, by definition of the Möbius function, only the subgroups of squarefree index contribute. We separate into the sums over the subgroups  $H$  of  $C$  of index divisible by  $p$ , and subgroups  $H$  of index not divisible by  $p$ . There is a bijection between these two sets given by sending a subgroup  $H$  of index not divisible by  $p$  to  $pH$ . Thus,

$$\begin{aligned} v_p(C_{\theta_G}(\mathbb{1}_G)) &= -v_p(|G|) + \sum_{\substack{C \leq G \\ p \nmid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{v_p(|H|)}{|G:H|} \mu(|C:H|) + \sum_{\substack{C \leq G \\ p \nmid |C|}} \sum_{\substack{H \leq C \\ p \mid |C:H|}} \frac{v_p(|pH|)}{|G:pH|} \mu(|C:pH|) \\ &= -v_p(|G|) + \sum_{\substack{C \leq G \\ p \nmid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{v_p(|H|)}{|G:H|} \mu(|C:H|) - \sum_{\substack{C \leq G \\ p \nmid |C|}} \sum_{\substack{H \leq C \\ p \mid |C:H|}} \frac{v_p(|H|) - 1}{p|G:H|} \mu(|C:H|) \\ &= -v_p(|G|) + \sum_{\substack{C \leq G \\ p \nmid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \left( v_p(|H|) \cdot \frac{p-1}{p} \cdot \frac{\mu(|C:H|)}{|G:H|} + \frac{\mu(|C:H|)}{p|G:H|} \right) \\ &= -v_p(|G|) + \underbrace{\sum_{\substack{C \leq G \\ p \nmid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} v_p(|H|) \cdot \frac{p-1}{p} \cdot \frac{\mu(|C:H|)}{|G:H|}}_{(\dagger)} + \underbrace{\sum_{\substack{C \leq G \\ p \nmid |C|}} \sum_{\substack{H \leq C \\ p \mid |C:H|}} \frac{\mu(|C:H|)}{p|G:H|}}_{(\star)} \end{aligned}$$

We claim that  $(\star)$  is equal to  $\frac{1}{|G|} \sum_{g \in G} \frac{1_p(g)}{p-1}$  and  $(\dagger)$  is equal to  $\frac{1}{|G|} \sum_{g \in G} v_p(|g|)$ . To see this observe that if  $f: G \rightarrow \mathbb{C}$  is any map of sets, constant on elements  $g \in G$  for which  $v_p(|g|)$  is equal, then a standard Möbius function argument shows that

$$\sum_{C \leq G} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{p-1}{p} \frac{\mu(|C:H|)}{|G:H|} f(h) = \frac{1}{|G|} \sum_{g \in G} f(g),$$

where on the right hand side  $h$  denotes any generator of  $H$ . Moreover, if we exclude on the left hand side those subgroups  $C \leq G$  with  $p \nmid |C|$  we find

$$\sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{p-1}{p} \frac{\mu(|C:H|)}{|G:H|} f(g) = \frac{1}{|G|} \sum_{\substack{g \in G \\ v_p(|g|) \geq 1}} f(g).$$

Setting  $f(g) = v_p(|g|)$  gives

$$\sum_{\substack{C \leq G \\ p \mid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} v_p(|H|) \cdot \frac{p-1}{p} \cdot \frac{\mu(|C:H|)}{|G:H|} = \frac{1}{|G|} \sum_{g \in G} v_p(|g|),$$



whilst taking  $f(g) = \frac{1}{p-1}$  shows that

$$\begin{aligned} \sum_{\substack{C \leq G \\ p \nmid |C|}} \sum_{\substack{H \leq C \\ p \nmid |C:H|}} \frac{\mu(|C:H|)}{p|G:H|} &= \frac{1}{|G|} \sum_{\substack{g \in G \\ v_p(|g|) \geq 1}} \frac{1}{p-1} \\ &= \frac{1}{|G|} \sum_{g \in G} \frac{\mathbb{1}_p(g)}{p-1}. \end{aligned}$$

In conclusion,

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = v_p(|G|) + \frac{1}{|G|} \sum_{g \in G} v_p(|g|) + \frac{1}{|G|} \sum_{g \in G} \frac{\mathbb{1}_p(g)}{p-1}.$$

□

We shall see that the value of (9) is less than or equal to zero for all groups  $G$ . Theorem 6.1 thus gives a numerical characterisation of groups for which all  $p$ -hypo-elementary subgroups are cyclic:

**COROLLARY 6.13** *Let  $G$  be any finite group and  $p$  a prime. Then,  $G$  contains no non-cyclic  $p$ -hypo-elementary subgroups if and only if*

$$\frac{1}{|G|} \sum_{g \in G} v_p(|g|) + \frac{1}{|G|} \sum_{g \in G} \frac{\mathbb{1}_p(g)}{p-1} = v_p(|G|).$$

The reverse direction whilst a consequence of the argument given in Remark 6.2 is already somewhat non-obvious.

**REMARK 6.14** Suppose that the Sylow  $p$ -subgroup of  $G$  is non-cyclic. Let  $d = v_p(|G|)$ , as  $G$  contains no elements of order  $p^d$ , for any  $g \in G$   $v_p(|g|) \leq d-1$  and we may crudely bound

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in G} \left( v_p(|g|) + \frac{\mathbb{1}_p(g)}{p-1} \right) &\leq \frac{1}{|G|} \sum_{g \in G} \left( d-1 + \frac{1}{p-1} \right) \\ &< d. \end{aligned}$$

Applying (9) gives

$$v_p(C_{\theta_G}(\mathbb{1}_G)) < 0.$$

The case of cyclic Sylow  $p$ -subgroup requires considerably more care.

## 6.2 Average $p$ -orders of elements of groups with cyclic Sylow subgroup

In this section, we complete the proof of Theorem 6.1 by explicit calculation of  $v_p(C_{\theta_G}(\mathbb{1}_G))$  via (9) for groups with cyclic Sylow  $p$  subgroups. In the process, for such groups, we obtain average values of the indicator function  $\mathbb{1}_p(g)$ , and of the valuation function  $v_p(|g|)$  in terms of elementary invariants.

**NOTATION 6.15** Fix a single prime  $p$  throughout. Let  $\mathcal{P}(G, k)$  denote the number of elements of a given finite group  $G$  whose order is divisible by  $p^k$ .

LEMMA 6.16 Let  $G$  be any group with cyclic Sylow  $p$ -subgroup of order  $p^r$ . Then for any  $1 \leq k \leq r$ ,

$$\mathcal{P}(G, k) = \left( \frac{p^{r-k+1} - 1}{p^{r-k+1}} \right) \frac{|G||Z_G(Q)|}{|N_G(Q)|},$$

where  $Q$  denotes any choice of non-trivial  $p$ -subgroup of  $G$ . If  $k = 0$ , then  $\mathcal{P}(G, k) = |G|$  and if  $k > r$ , then  $\mathcal{P}(G, k) = 0$ .

We split the proof into four intermediate claims. Firstly, the ratio  $|Z_G(Q)|/|N_G(Q)|$  is independent of the choice of  $Q$ :

CLAIM 1 If  $G$  is any finite group and  $p$  a prime such that  $G$  has cyclic Sylow  $p$ -subgroup, then, as  $Q$  runs over non-trivial  $p$ -subgroups,  $|N_G(Q)|/|Z_G(Q)|$  is constant.

*Proof.* For such a group all  $p$ -subgroups of the same order are conjugate. If  $Q, Q'$  are conjugate  $p$ -subgroups their normalisers and centralisers are related by conjugation and so the above ratio is constant. Thus we need just show that if  $P$  is a subgroup of order  $p^e$ ,  $e \geq 2$ , and  $Q$  its unique subgroup of order  $p^{e-1}$  then

$$\frac{|N_G(P)|}{|Z_G(P)|} = \frac{|N_G(Q)|}{|Z_G(Q)|}. \quad (10)$$

First note that  $N_G(P) \cap Z_G(Q) = Z_G(P)$ . This is because both sides contain  $P$ , but the coprime to  $p$ -part of  $\text{Aut}(P)$  is canonically isomorphic to the coprime to  $p$ -part of  $\text{Aut}(Q)$  (both are cyclic of order  $p-1$ ). In other words, within  $N_G(P)$ , to centralise  $Q$  is to centralise  $P$ . As a result, there is an inclusion

$$N_G(P)/Z_G(P) \hookrightarrow N_G(Q)/Z_G(Q),$$

and to prove (10) we must show that  $N_G(Q) = N_G(P)Z_G(Q)$ . Indeed, as all terms are contained in  $N_G(Q)$ , we may assume that  $Q \trianglelefteq G$ . Each choice of subgroup of order  $p^e$  (i.e. conjugate of  $P$ ) must centralise  $Q$ , its unique subgroup of order  $p^{e-1}$ , thus  $\bigcup_{g \in G/N_G(P)} P^g \subseteq Z_G(Q)$ . In particular,  $Z_G(Q)$  contains a representative of each coset of  $G/N_G(P)$  and so  $N_G(P)Z_G(Q) = G$ . And in general,  $N_G(P)Z_G(Q) = N_G(Q)$ .  $\square$

We find that to prove the formula for fixed  $k$  we may restrict to those groups with a central  $C_{p^k}$  subgroup.

CLAIM 2 Let  $G$  be any finite group and  $p$  any prime, then, for any  $k \geq 1$ , all elements of order divisible by  $p^k$  are contained within  $\bigcup_Q Z_G(Q)$  as  $Q$  runs through subgroups of  $G$  isomorphic to  $C_{p^k}$ . Moreover, if all  $Q$  are conjugate then

$$\mathcal{P}(G, k) = |G : N_G(Q)| \mathcal{P}(Z_G(Q), k),$$

for any choice of  $Q \cong C_{p^k}$ .

*Proof.* Let  $g \in G$  and  $v_p(|g|) \geq k$ . Then,  $g$  centralises the subgroup of  $\langle g \rangle$  isomorphic to  $C_{p^k}$ . So  $g$  is contained in  $\bigcup_Q Z_G(Q)$ , the union of the centralisers of all  $C_{p^k}$  subgroups of  $G$ . Now suppose all the  $C_{p^k}$ -subgroups are conjugate. Let  $Q, Q'$  be two  $C_{p^k}$ -subgroups of  $G$  and suppose  $Z_G(Q) \cap Z_G(Q')$  contains a subgroup  $T$  isomorphic to  $C_{p^k}$ . Then  $T = Q = Q'$  as  $Q, Q'$  are the unique  $C_{p^k}$ -subgroups of  $Z_G(Q), Z_G(Q')$  respectively. So  $\mathcal{P}(Z_G(Q) \cap Z_G(Q'), k) = 0$  and

$$\mathcal{P}(G, k) = \sum_Q \mathcal{P}(Z_G(Q), k) = |G : N_G(Q)| \mathcal{P}(Z_G(Q), k).$$

$\square$

As a basis for induction we show:

CLAIM 3 Let  $G$  be any group and  $C_p$  a subgroup of order  $p$  that is contained in the centre. Then

$$\mathcal{P}(G, 1) = \mathcal{P}(G/C_p, 1) + \frac{p-1}{p} \cdot |G|.$$

*Proof.* Consider the sequence

$$1 \rightarrow C_p \rightarrow G \xrightarrow{q} G/C_p \rightarrow 1,$$

and let  $h$  run over elements of  $G/C_p$ . First assume that  $h$  has order not divisible by  $p$ . As  $C_p \leq Z(G)$ , the preimage of  $\langle h \rangle$  is isomorphic to  $C_p \times C_{|h|}$  on which  $q$  is projection onto the second factor. Thus,  $q^{-1}(h)$  contains precisely  $p-1$  elements of order  $p$ .

Otherwise, if  $h$  has order divisible by  $p$ , then all elements of  $q^{-1}(h)$  have order divisible by  $p$ . As a result

$$\mathcal{P}(G, 1) = p\mathcal{P}(G/C_p, 1) + (p-1)(|G/C_p| - \mathcal{P}(G/C_p, 1)),$$

giving the stated formula.  $\square$

The inductive step is given by:

CLAIM 4 Let  $G$  be any group with cyclic Sylow  $p$ -subgroup and containing a central subgroup of order  $p^k$  with  $k \geq 2$ . Then

$$\mathcal{P}(G, k) = p\mathcal{P}(G/C_p, k-1).$$

*Proof.* As in Claim 3, let  $Q$  be a subgroup isomorphic to  $C_p$  and consider the sequence

$$1 \rightarrow Q \rightarrow G \xrightarrow{q} G/Q \rightarrow 1.$$

Running over elements  $h \in G/Q$ , we find that if  $p^k$  divides  $|h|$ , then all  $p$  preimages have order divisible by  $p^k$  and conversely if  $p^{k-1} \nmid |h|$ , then none do.

Now assume that  $p^{k-1}$  is the maximal power of  $p$  dividing  $|h|$ . Then,  $H := q^{-1}(\langle h \rangle)$  is a subgroup of  $G$  with Sylow  $p$ -subgroup of order  $p^k$ . Thus,  $H$  must be of the form  $C_{p^k} \times A$  with  $p$  not dividing the order of  $A$ . Via this description  $q$  is the quotient  $C_{p^k} \times A \rightarrow C_{p^k}/C_p \times A$ . So, as  $h$  has order divisible by  $p$ , all elements in the fibre of  $h$  are divisible by  $p$ . In conclusion,

$$\mathcal{P}(G, k) = p\mathcal{P}(G/C, k-1).$$

$\square$

*Proof of Lemma 6.16.* We first show the formula when  $k = 1$ . Applying Claim 2, we assume that  $G$  contains a central subgroup  $Q$  isomorphic to  $C_p$ . The formula trivially holds when  $r = 0$  and when  $r = 1$  by Claim 3. Now assume  $r \geq 2$ . We wish to show that

$$\mathcal{P}(G, 1) = \left( \frac{p^r - 1}{p^r} \right) |G|.$$

Applying Claim 3 and the inductive hypothesis,

$$\begin{aligned} \mathcal{P}(G, 1) &= \mathcal{P}(G/Q, 1) + \left( \frac{p-1}{p} \right) |G| \\ &= \left( \frac{p^{r-1} - 1}{p^{r-1}} \right) \frac{|G/Q| \cdot |Z_{G/Q}(Q')|}{|N_{G/Q}(Q')|} + \left( \frac{p-1}{p} \right) |G|, \end{aligned}$$

where  $Q'$  is a choice of  $C_p$  subgroup of  $G/Q$ . Let  $P$  denote the preimage of  $Q'$  in  $G$ . Recall, for a chain of subgroups  $A \geq B \geq C$  with  $C \leq A$ , then  $N_A(B)/C \cong N_{A/C}(B/C)$ . Moreover if  $C \subseteq Z(A)$ , then  $Z_A(B)/C = Z_{A/C}(B/C)$ . Thus,  $|G/Q : N_{G/Q}(Q')| = |G : N_G(P)|$  and  $|Z_{G/Q}(Q')| = \frac{1}{p}|Z_G(P)|$ . So

$$\begin{aligned} &= \left( \frac{p^{r-1} - 1}{p^{r-1}} \right) \frac{|G| \cdot |Z_G(P)|}{|N_G(P)| \cdot p} \\ &= \left( \frac{p-1}{p} + \frac{p^{r-1} - 1}{p^r} \right) |G| \\ &= \left( \frac{p^r - 1}{p^r} \right) |G| \end{aligned}$$

as required, where we used the independence asserted in Claim 1 to show

$$\frac{|Z_G(P)|}{|N_G(P)|} = \frac{|Z_G(Q)|}{|N_G(Q)|} = 1.$$

Thus, the formula holds when  $k = 1$ .

Now assume  $k > 1$  and that the formula holds for all groups and indices  $\ell < k$ . By Claim 1, we are reduced to verifying the formula for groups with a central subgroup  $Q$  isomorphic to  $C_{p^k}$ .

Fix a  $Q$  subgroup of isomorphic to  $C_p$  of such a  $G$ . Applying Claim 4,

$$\begin{aligned} \mathcal{P}(G, k) &= p\mathcal{P}(G/Q, k-1) \\ &= p \left( \frac{p^{(r-1)-(k-1)+1} - 1}{p^{(r-1)-(k-1)+1}} \right) \frac{|G/Q| \cdot |G/Q|}{|G/Q|} \\ &= \left( \frac{p^{r-k+1} - 1}{p^{r-k+1}} \right) |G| \end{aligned}$$

which is the required formula.  $\square$

*Proof of Theorem 6.1.* By Remarks 6.2 and Corollary 6.9, we need only prove that if  $G$  has a non-cyclic  $p$ -hypo-elementary subgroup  $v_p(C_{\theta_G}(\mathbb{1}_G)) < 0$ . Whilst, by Remark 6.14, we may assume  $G$  has cyclic Sylow  $p$ -subgroup.

Applying Lemma 6.12, we want to show for such  $G$  that

$$\frac{1}{|G|} \sum_{g \in G} v_p(|g|) + \frac{1}{|G|} \sum_{g \in G} \frac{\mathbb{1}_p(g)}{p-1} \leq v_p(|G|),$$

with equality if and only if all  $p$ -hypo-elementary subgroups of  $G$  are cyclic. Lemma 6.16 shows that if  $G$  has Sylow  $p$ -subgroup isomorphic to  $C_{p^r}$ , then

$$\frac{1}{|G|} \sum_{g \in G} v_p(|g|) = \sum_{i=1}^r \left( \frac{p^{r-k+1} - 1}{p^{r-k+1}} \right) \frac{|G||Z_G(Q)|}{|N_G(Q)|}$$

and

$$\frac{1}{|G|} \sum_{g \in G} \frac{\mathbb{1}_p(g)}{p-1} = \left( \frac{p^r - 1}{(p-1)p^r} \right) \cdot \frac{|G||Z_G(Q)|}{|N_G(Q)|},$$

where  $Q$  denotes any choice of subgroup of  $G$  isomorphic to  $C_p$ . Thus,

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in G} v_p(|g|) + \frac{1}{|G|} \sum_{g \in G} \frac{\mathbb{1}_p(g)}{p-1} &= \left( \sum_{i=1}^r \frac{p^{r-i+1} - 1}{p^{r-i+1}} + \frac{p^r - 1}{(p-1)p^r} \right) \frac{|Z_G(Q)|}{|N_G(Q)|} \\ &= \left( \sum_{i=1}^r \frac{p^{r-i+1} - 1}{p^{r-i+1}} + \sum_{i=1}^r \frac{1}{p^{r-i+1}} \right) \frac{|Z_G(Q)|}{|N_G(Q)|} \\ &= r \cdot \frac{|Z_G(Q)|}{|N_G(Q)|}. \end{aligned}$$

So that

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -r \cdot \left( 1 - \frac{|Z_G(Q)|}{|N_G(Q)|} \right)$$

whenever  $G$  has cyclic Sylow  $p$ -subgroup. Finally, note that a group with cyclic Sylow  $p$ -subgroup has no non-cyclic  $p$ -hypo-elementary groups if and only if all subgroups of order  $pq$  with  $q$  a prime distinct to  $p$  are isomorphic to  $C_p \times C_q$ . The latter holds if and only if the normaliser of each  $C_p$  subgroup is equal to its centraliser. So indeed  $v_p(C_{\theta_G}(\mathbb{1}_G)) < 0 \iff G$  contains a non-cyclic Sylow  $p$ -subgroup, otherwise it is zero.  $\square$

It is worth stating that during the proof we derived the following corollary:

**COROLLARY 6.17** *For any finite group  $G$  and prime  $p$  such that the Sylow  $p$ -subgroups of  $G$  are cyclic,*

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -r \cdot \left( 1 - \frac{|Z_G(Q)|}{|N_G(Q)|} \right).$$

Here  $Q$  denotes any choice of non-trivial  $p$ -subgroup of  $G$  unless  $p \nmid |G|$  in which case  $Q = \{1\}$  and  $v_p(C_{\theta_G}(\mathbb{1}_G)) = 0$ .

When  $G$  doesn't have cyclic Sylow  $p$ -subgroup, we can only say that  $v_p(C_{\theta_G}(\mathbb{1}_G)) \leq -\frac{p}{|G|}$ .

**EXAMPLE 6.18** Let  $G$  be a  $p$ -hypo-elementary group with non-trivial cyclic Sylow  $p$ -subgroup. Then  $G$  is of the form  $C_{p^r} \rtimes C_n$  with  $(p, n) = 1$ . Let  $S$  denote the kernel of the map  $C_n \rightarrow \text{Aut}(C_{p^r})$  defining the semi-direct product and  $s = |S|$ . Then,

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -r \left( 1 - \frac{s}{n} \right),$$

as the centraliser of  $C_{p^r}$  is  $C_{p^r} \times S \leq G$  (the action is trivial) and  $C_{p^r} \trianglelefteq G$ .

We can also verify this directly. In Example, 2.19 we saw that the Artin relation of such a  $G$  is given by

$$\theta_G = [C_{p^r} \rtimes C_n] - [C_n] + \frac{s}{n}[S] - \frac{s}{n}[C_{p^r} \times S].$$

So applying formula (4)

$$C_{\theta_G}(\mathbb{1}_G) = \frac{\left( \frac{1}{|G|} \right) \cdot \left( \frac{1}{|S|} \right)^{\frac{s}{n}}}{\left( \frac{1}{|C_n|} \right) \left( \frac{1}{|C_{p^r} \times S|} \right)^{\frac{s}{n}}},$$

so that indeed

$$v_p(C_{\theta_G}(\mathbb{1}_G)) = -r + \frac{s}{n} \cdot r.$$

## 7 Examples

### 7.1 $D_{2p}$

Let  $G = D_{2p}$  be the dihedral group of order  $2p$  for  $p$  an odd prime. In this section, we explicitly demonstrate Theorem 5.4 for such  $G$  and all primes  $\ell$ . More precisely, we calculate the matrix representing  $\kappa \oplus \text{Yak}$  of Remark 5.5. This is possible as  $D_{2p}$  is one of the few groups for which the isomorphism classes of all indecomposable  $\mathbb{Z}_\ell[G]$ -lattices have been classified, even when  $\ell$  divides  $|G|$ .

We first describe a basis of  $A(\mathbb{Z}_\ell[G])_\mathbb{Q}$ . From Example 2.29, a basis of  $A(\mathbb{Z}_\ell[G], \text{triv})_\mathbb{Q}$  is given by

$$S = \begin{cases} \mathbb{1}_{\{1\}} \uparrow^G, \mathbb{1}_{C_2} \uparrow^G, \mathbb{1}_{C_p} \uparrow^G & \ell \neq p \\ \mathbb{1}_{\{1\}} \uparrow^G, \mathbb{1}_{C_2} \uparrow^G, \mathbb{1}_{C_p} \uparrow^G, \mathbb{1}_G & \ell = p \end{cases}.$$

When  $\ell \neq 2, p$ , all  $\mathbb{Z}_\ell[G]$ -lattices have trivial source so that  $S$  forms a basis of  $A(\mathbb{Z}_\ell[G])_\mathbb{Q}$ .

When  $\ell = 2$ , as  $N_{D_{2p}}(C_2) = C_2$ , the Yakovlev diagram for a module  $M$  consists of  $H^1(C_2, M)$  as an abelian group. So, applying Theorem 5.2, any  $\mathbb{Z}_{(2)}[D_{2p}]$ -lattice  $M$  for which  $H^1(C_2, M) \cong \mathbb{Z}/2\mathbb{Z}$  will extend  $S$  to a basis of  $A_\mathbb{Q}(\mathbb{Z}_{(2)}[G])$ . The sign representation  $\epsilon$ , that is the non-trivial one dimensional irreducible lifted from  $\mathbb{Z}_{(2)}[D_{2p}/C_p]$ , is one such module.

When  $\ell = p$ , the Yakovlev diagram of a  $\mathbb{Z}_{(p)}[G]$ -lattice  $M$  consists of  $H^1(C_p, M)$  as a  $\mathbb{F}_p[D_{2p}/C_p]$ -module. Since  $\text{char}(\mathbb{F}_p) \neq 2$ , there are two irreducible  $\mathbb{F}_p[D_{2p}/C_p]$ -modules, both one dimensional, one with trivial action and one without. So any two lattices whose cohomology exhibits these modules will extend  $S$  to a basis of  $A(\mathbb{Z}_{(p)}[G])_\mathbb{Q}$ . If  $\rho$  denotes the  $(2p-2)$ -dimensional irreducible  $\mathbb{Q}[G]$ -representation, then there are two non-isomorphic  $\mathbb{Z}_{(p)}[G]$ -sublattices  $A, A'$  contained in  $\rho$ , with  $H^1(C_p, A), H^1(C_p, A') \cong \mathbb{Z}/p\mathbb{Z}$  as abelian groups, but the former having trivial  $D_{2p}/C_p$  action and the latter non-trivial action. These modules are explicitly constructed in [Lee64].

In conclusion,

$$\dim(A(\mathbb{Z}_\ell[G])_\mathbb{Q}) = \begin{cases} 3 & \ell \neq 2, p \\ 4 & \ell = 2 \\ 6 & \ell = p \end{cases}.$$

with a basis  $S'$  given by

$$S' = \begin{cases} \mathbb{1}_{\{1\}} \uparrow^G, \mathbb{1}_{C_2} \uparrow^G, \mathbb{1}_{C_p} \uparrow^G & \ell \neq 2, p \\ \mathbb{1}_{\{1\}} \uparrow^G, \mathbb{1}_{C_2} \uparrow^G, \mathbb{1}_{C_p} \uparrow^G, \epsilon & \ell = 2 \\ \mathbb{1}_{\{1\}} \uparrow^G, \mathbb{1}_{C_2} \uparrow^G, \mathbb{1}_{C_p} \uparrow^G, \mathbb{1}_G, A, A' & \ell = p \end{cases}.$$

The matrix representing  $\kappa \oplus \text{Yak}$  is then

$$\begin{aligned}
 & \langle -, \mathbb{1}_{\uparrow_H^G} \rangle \begin{pmatrix} \mathbb{1}_{\uparrow_{\{1\}}^G} & \mathbb{1}_{\uparrow_{C_2}^G} & \mathbb{1}_{\uparrow_{C_p}^G} \\ 2p & p & 2 \\ p & (p+1)/2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \quad \text{if } \ell \neq 2, p. \\
 & \langle -, \mathbb{1}_{\uparrow_H^G} \rangle \begin{pmatrix} \mathbb{1}_{\uparrow_{\{1\}}^G} & \mathbb{1}_{\uparrow_{C_2}^G} & \mathbb{1}_{\uparrow_{C_p}^G} & \epsilon \\ 2p & p & 2 & 1 \\ p & (p+1)/2 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ \hline 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{if } \ell = 2 \\
 & \text{Yak}(-) \begin{pmatrix} \mathbb{1}_{\uparrow_{\{1\}}^G} & \mathbb{1}_{\uparrow_{C_2}^G} & \mathbb{1}_{\uparrow_{C_p}^G} & \mathbb{1}_G & A & A' \\ 2p & p & 2 & 1 & 2 & 2 \\ p & (p+1)/2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{if } \ell = p
 \end{aligned}$$

For the above matrices, when  $\ell = 2$ , the basis of  $\mathcal{C}$  is taken to be  $\mathbb{Z}/2\mathbb{Z}$ , and when  $\ell = p$ , the basis is given by  $\mathbb{Z}/p\mathbb{Z}$  with both its trivial and non-trivial  $C_2 \cong D_{2p}/C_p$ -actions. The calculations of  $v_p(C_{\theta_G}(A)), v_p(C_{\theta_G}(A'))$  can be found in [Bar12, Thm. 4.4].

REMARK 7.1 For  $p \leq 67$ , a  $\mathbb{Z}[D_{2p}]$ -lattice is determined by its localisation at the primes 2,  $p$  (see [Bar12, Ex. 6.3]). So, by applying Theorem 5.4 at both primes we obtain a finite list of data which specifies the isomorphism class of an arbitrary  $\mathbb{Z}[D_{2p}]$ -lattice.

## 7.2 Groups with degenerate permutation pairing

EXAMPLE 7.2 Let  $G = C_3 \times C_3 \times S_3$ , a 3-hypo-elementary group. Up to conjugacy  $G$  has 17 subgroups, but the permutation pairing of Construction 3.8 is degenerate and has rank 16.

In this section, we define a canonical Brauer relation  $\theta_{\Sigma, G}$  which is non-zero for any non-cyclic group  $G$ . When  $G = C_3 \times C_3 \times S_3$ , then  $\theta_{\Sigma, G}$  generates the kernel of the permutation pairing.

NOTATION 7.3 • Let  $G$  be any finite group and  $\Sigma$  denote the set of subgroups of  $G$ , a partially ordered set with respect to containment. Let  $\mu_{\Sigma}: \Sigma \rightarrow \mathbb{Z}$  denote the Möbius function on  $\Sigma$ , i.e. the unique function for which

$$\mu_{\Sigma}(G) = 1$$

and

$$\sum_{H' \geq H} \mu_{\Sigma}(H') = 0$$

for all non maximal  $H$ .

- Set

$$\theta_{\Sigma, G} = \sum_{H \in \Sigma} \frac{\mu_{\Sigma}(H)}{|G:H|} [H] \in B(G).$$



- For an element  $\theta \in B(G)$  and  $K \leq G$ , let  $\theta^K$  denote the number of fixed points of  $\theta$  under  $K$ , i.e. if  $\theta = \sum_H \alpha_H [H]$  then  $\theta^K = \sum \alpha_H \#([H]^K)$ .

LEMMA 7.4 For any  $K \leq G$ ,  $(\theta_{\Sigma, G})^K = \sum_{H \geq K} \mu_S(H)$ .

*Proof.* Since both  $\#[H]^K$  and  $\mu_S(H)$  are constant under replacing  $H$  with a conjugate, we have

$$\begin{aligned} \sum_{H \leq G} \frac{\mu_{\Sigma}(H)}{|G : H|} \#[H]^K &= \sum_{\substack{H \leq_G G \\ H \in S}} \frac{|H| \mu_{\Sigma}(H)}{|N_G(H)|} \#[H]^K \\ &= \sum_{H \leq_G G} \frac{|H| \mu_{\Sigma}(H)}{|N_G(H)|} |\{g \in G/H \mid K^g \leq H\}| \\ &= \sum_{H \leq_G G} \frac{|H| \mu_{\Sigma}(H)}{|N_G(H)|} |\{g \in G/H \mid K \leq H^g\}| \\ &= \sum_{H \leq_G G} \frac{|H| \mu_{\Sigma}(H)}{|N_G(H)|} \cdot \frac{|N_G(H)|}{|H|} |\{g \in G/N_G(H) \mid K \leq H^g\}| \\ &= \sum_{H \geq K} \mu_{\Sigma}(H) \end{aligned}$$

□

COROLLARY 7.5 For any finite group  $G$ ,

- i)  $\theta_{\Sigma, G}$  is a Brauer relation in characteristic zero if and only if  $G$  is non-cyclic.
- ii)  $\theta_{\Sigma, G}$  is a Brauer relation in characteristic  $p$  if and only if  $G$  is non- $p$ -hypo-elementary.
- iii)  $\theta_{\Sigma, G} \downarrow_H$  is zero for all proper subgroups  $H$ .

*Proof.* We prove i). An element of  $B(G)$  is a relation in characteristic zero if and only if the number of its fixed points under all cyclic subgroups is zero (Lemma A.8). By the lemma,  $\theta_{\Sigma, G}$  is a relation in characteristic zero if and only if

$$\sum_{H \geq C} \mu_{\Sigma}(H) = 0$$

for all cyclic subgroups  $C$ . By the definition of  $\mu_{\Sigma}$ , this is true if and only if  $G$  is not itself cyclic.

The argument for ii) is identical instead using that elements of  $B(G)$  are relations in characteristic  $p$  if and only if they the number of fixed points under all  $p$ -hypo-elementary subgroups is zero (see Lemma A.8).

For iii), simply note that an element of  $B(H)$  is zero if and only if its fixed points under all subgroups is zero (Lemma A.8 i)). But  $(\theta_{\Sigma, G} \downarrow_H)^K = (\theta_{\Sigma, G})^K$ , which by the lemma vanishes for all proper subgroups  $K < G$ . □

REMARK 7.6 By Lemma 2.33 iv),  $\theta_{\Sigma, G}$  automatically vanishes on all permutation modules other than possibly the trivial representation. In Example 7.2,  $C_3 \times C_3 \times S_3$  is a group for which in addition  $\theta_{\Sigma, G}(\mathbb{1}_G) = 1$ .

EXAMPLE 7.7 Let  $G = (C_p \times C_p) \rtimes C_q$  with  $p, q$  odd primes and  $q = 2p + 1$  and where  $C_q$  acts by scaling on  $C_p \times C_p$ . Write  $\alpha_H = \mu_{\Sigma}(H)/|G : H|$  so that  $\theta_{\Sigma, G} = \sum_{H \leq G} \alpha_H [H]$ . Then, the  $\alpha_H$  for each conjugacy class are given in the following table:

	1	$C_q$	$C_p$	$C_p$	$C_p$	$C_p$	$C_p \rtimes C_q$	$C_p \rtimes C_q$	$C_p \times C_p$	$G$
#conjugates	1	$p^2$	1	1	$q$	$q$	$p$	$p$	1	1
$\mu_\Sigma(H)$	$-p^2$	1	$p$	$p$	0	0	-1	-1	-1	1
$\alpha_H$	$-1/q$	$1/p^2$	$1/q$	$1/q$	0	0	$-1/p$	$-1/p$	$-1/q$	1

And we find  $\theta_{\Sigma,G}(\mathbb{1}_G) = 1$ . Thus, the relation  $\theta_{\Sigma,G}$  is trivial on all permutation representations. As  $G$  is  $p$ -hypo-elementary,  $\theta_{\Sigma,G}$  is not a  $p$ -relation and the permutation pairing of Construction 3.8 is degenerate.

It is not clear to the author if, for any of the above groups that have degenerate permutation pairing, there exists a module  $M$  for which  $\theta_{\Sigma,G}(M) \neq 1$ . Necessarily, such an  $M$  must not be trivial source (Theorem 2.24) or induced from a proper subgroup (Lemma 2.33).

## A Induction theorems and Brauer relations

The principal aim of this appendix is to provide a proof of Theorem 2.24, which, whilst well known, we have been unable to provide a reference for. All notation can be found in Sections 2.1, 2.3.

**DEFINITION A.1** Let  $G$  be a finite group and fix a prime  $p$ . Let  $|G| = p^k m$  with  $(p, m) = 1$ , and set  $K = \mathbb{Q}_p(\zeta_m)$  to be the extension obtained by adjoining the  $m^{\text{th}}$  roots of unity, finally write  $\mathcal{O}_K$  for the valuation ring of  $K$ . A *species* is a ring homomorphism  $A(\mathcal{O}_K[G], \text{triv}) \rightarrow \mathbb{C}$ .

**CONSTRUCTION A.2** For  $H$  a subgroup of  $G$ , we say that an indecomposable trivial source  $\mathcal{O}_K[G]$ -lattice  $M$  has vertex  $H$  if  $M$  is a direct summand of  $\mathbb{1}_H^G$  but not of  $\mathbb{1}_{H'}^G$  for any  $H' \subsetneq H$ . The set of vertices of  $M$  form a conjugacy class of subgroups and only  $p$ -groups appear as vertices [Ben98, Prop. 3.10.2]. Any lattice  $M$  can be decomposed into summands, one for each conjugacy class of subgroups, each of whose indecomposable constituents has vertices lying within that conjugacy class. Moreover, if  $N$  is a  $\mathbb{Z}_{(p)}[G]$ -lattice, the decomposition of  $N \otimes \mathcal{O}_K$  is defined over  $\mathbb{Z}_{(p)}$ .

**NOTATION A.3** If  $P$  is a  $p$ -subgroup of  $G$ , then we write  $A(\mathcal{O}_K[G], \text{triv})^{\leq P}$  (resp.  $A(\mathcal{O}_K[G], \text{triv})^{< P}$ ) for subspace of the trivial source ring spanned by indecomposables with vertex contained in (resp. strictly contained in)  $P$ .

**CONSTRUCTION A.4** Consider pairs  $(P, g)$ , where  $P \leq G$  is a  $p$ -group and  $g$  an element of  $N_G(P)$  of order coprime to  $p$ , up to simultaneous conjugacy. To any such pair we may associate a species  $t_{(P,g)}$  as follows. There are isomorphisms

$$\begin{array}{c} A(\mathcal{O}_K[G], \text{triv})^{\leq P} / A(\mathcal{O}_K[G], \text{triv})^{< P} \xrightarrow{\sim} A(\mathcal{O}_K[N_G(P)], \text{triv})^{\leq P} / A(\mathcal{O}_K[N_G(P)], \text{triv})^{< P} \\ \downarrow \sim \\ A(\mathcal{O}_K[N_G(P)/P], \text{triv})^{\leq \{1\}} \end{array}$$

Here the first map is induced by restriction and is an isomorphism by Green correspondence [CR94b, Thm. 81.35]. The second map sends a lattice  $M$  to its vertex  $P$  summand, which, as  $M$  is trivial source, can be considered as an  $N_G(P)/P$ -module. We define  $t_{(P,g)}: A(\mathbb{Z}_{(p)}[G], \text{triv}) \rightarrow \mathbb{C}$  to be the postcomposition with  $\text{tr}(g \mid (-))$ , i.e.  $t_{(P,g)}(M)$  is the trace of  $g$  acting on  $N$ .

**REMARK A.5** This definition is equivalent to that of [Ben98, Sec. 5.5]. This follows from the following claim: For any  $H$  with  $P \leq H \leq N_G(P)$ , and trivial source  $\mathcal{O}_K[N_G(P)]$ -module  $M$  with

vertex  $P$ , then the indecomposable summands of  $M \downarrow_H$  remain of vertex  $P$ . To see this we may assume  $P = H$ , then any indecomposable summand of  $M \downarrow_P$  is a summand of  $\mathbb{1}_P \uparrow^{N_G(P)} \downarrow_P$  and so by Mackey must be the trivial module. But the trivial module of a  $P$ -group has vertex  $P$  (see [CR94b, Prop. 57.29]).

All species are of the form  $t_{(P,g)}$  for some  $(P, g)$ :

**THEOREM A.6** (Conlon [Ben98, Cor. 5.5.5]) *For any finite group  $G$  and  $K$  as above, there is an isomorphism*

$$\prod t_{(P,g)}: A(K[G], \text{triv})_{\mathbb{C}} \rightarrow \prod_{(H,g)} \mathbb{C}.$$

We can now prove the statement of Theorem 2.24:

**THEOREM A.7** *Let  $G$  be any finite group and  $M$  a trivial source  $\mathbb{Z}_{(p)}[G]$ -module. Then, there exist unique  $\alpha_H \in \mathbb{Q}$ , as  $H$  runs over conjugacy classes of  $p$ -hypo-elementary subgroups of  $G$ , such that*

$$M \cong \sum_{H \in \text{hyp}_p(G)} \alpha_H \mathbb{1}_H \uparrow^G.$$

*Proof.* We claim that for any pair  $(P, g)$  and trivial source  $\mathbb{Z}_{(p)}[G]$ -module  $M$ ,  $t_{(P,g)}(M)$  is invariant under replacing  $g$  with any other generator of  $\langle g \rangle$ . Indeed, there is an inclusion  $A(\mathbb{Z}_{(p)}[N_G(P)/P], \text{triv})^{\leq \{1\}} \hookrightarrow A(\mathbb{Q}[N_G(P)/P])$ , and for rational representations Artin's induction theorem 2.14 shows that characters are constant on elements which generate the same cyclic subgroup. Thus, two maps  $t_{(P,g)}: A(\mathbb{Z}_{(p)}[G], \text{triv}) \rightarrow \mathbb{C}$  are equal if each  $P, g$  together generate conjugate  $p$ -hypo-elementary groups. By Theorem A.6,  $\dim_{\mathbb{Q}} A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{Q}}$  is less than or equal to the number of conjugacy classes of  $p$ -hypo-elementary subgroups.

Write  $t_H$  for  $t_{(P,g)}$  with any choice of  $g$  such that  $\langle P, g \rangle = H$ . To conclude, we must show that for all  $p$ -hypo-elementary groups up to conjugacy,  $t_H: A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{C}} \rightarrow \mathbb{C}$  are linearly independent.

**CLAIM** *If  $T$  is any group, then the functions  $\text{tr}(g \mid (-)): A(\mathbb{Z}_{(p)}[T], \text{triv})_{\mathbb{C}}^{\leq \{1\}} \rightarrow \mathbb{C}$ , as  $g$  runs over elements of order coprime to  $p$  which generate non-conjugate subgroups, are linearly independent.*

To see this, note that for any cyclic group  $C$  of order coprime to  $p$ ,  $\mathbb{1}_C^T$  must have vertices contained in a  $p$ -subgroup of  $C$ , and so  $\mathbb{1}_C^T \in A(\mathbb{Z}_{(p)}[T], \text{triv})_{\mathbb{C}}^{\leq \{1\}}$ . But  $\text{tr}(g \mid \mathbb{1}_C^T) = [C]^{(g)}$ , so ordering the  $\langle g \rangle$  by decreasing size gives the desired independence.

Applying the claim, we find that the  $t_H$  with Sylow  $p$ -subgroup  $P$  are linearly independent as functions on  $A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{C}}^{\leq P} / A(\mathbb{Z}_{(p)}[G], \text{triv})_{\mathbb{C}}^{\leq P} \rightarrow \mathbb{C}$ . As  $t_H: A(\mathbb{Z}_{(p)}[G], \text{triv}) \rightarrow \mathbb{C}$  vanishes on all modules whose vertex is not conjugate to an overgroup of  $P$ , inductively we find that all  $t_H$  must be linearly independent.  $\square$

We provide a sample application of Theorem 2.24:

**LEMMA A.8** *Let  $G$  be a finite group and  $\sum_i [H_i] - \sum_j [K_j] \in B(G)$ . Then,*

- i)  $\sum_i [H_i] - \sum_j [K_j]$  is equal to zero in  $B(G)$  if and only if, for all subgroups  $T \leq G$ , the number of fixed points of  $\coprod_i (G/H_i)$  and  $\coprod_j (G/K_j)$  under  $T$  are equal,
- ii)  $\sum_i [H_i] - \sum_j [K_j]$  is Brauer relation in characteristic zero if and only if, for all cyclic subgroups  $T \leq G$ , the number of fixed points of  $\coprod_i (G/H_i)$  and  $\coprod_j (G/K_j)$  under  $T$  are equal,
- iii)  $\sum_i [H_i] - \sum_j [K_j]$  is Brauer relation in characteristic  $p$  if and only if, for all  $p$ -hypo-elementary subgroups  $T \leq G$ , the number of fixed points of  $\coprod_i (G/H_i)$  and  $\coprod_j (G/K_j)$  under  $T$  are equal.

## REFERENCES

*Proof.* For *i*), note that the fixed points of  $\sum_i [H_i]$  and  $\sum_j [K_j]$  under  $G$  are equal if and only if  $[G]$  occurs an equal number of times on both sides. Similarly, by sequentially considering subgroups ordered by decreasing size, we find  $\sum_i [H_i] = \sum_j [K_j]$  if and only if the fixed points under all subgroups are equal (cf. (1) p7).

For *iii*) consider the commutative diagram

$$\begin{array}{ccc} B(G) & \xrightarrow{b_{G,p}} & A(\mathbb{F}_p[G], \text{perm}) \\ \downarrow & & \downarrow \\ \prod_{H-p\text{-hypo}} B(H) & \xrightarrow{\sim_{b_{H,p}}} & \prod_{H-p\text{-hypo}} A(\mathbb{F}_p[H], \text{perm}) \end{array}$$

where the horizontal arrows send  $G$ -sets to their permutation representations and the vertical maps are given by restriction. The lower arrow is an inclusion by Theorem 2.24 (the passage to  $\mathbb{F}_p$  is Lemma 2.11). The injectivity of right hand map can be seen by writing  $M = \sum_{H \in \text{hyp}_p(G)} \mathbb{1}_H^G$  (Theorem 2.24) and sequentially restricting to  $p$ -hypo-elementary subgroups in a similar way to *i*). Brauer relations in characteristic  $p$  are by definition the elements of  $\ker b_{G,p}$  and thus are the elements of  $B(G)$  which lie in the kernel of restriction  $B(G) \rightarrow \prod_{H \in \text{hyp}_p(G)} B(H)$ . By *i*) these are the elements of  $B(G)$  whose fixed points under all  $p$ -hypo-elementary subgroups cancel.

The proof of *ii*) is identical instead using Artin's induction theorem 2.14.  $\square$

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